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Scheide, Diego

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Hrsg.: Leiter des Instituts für Mathematik
Weimarer Straße 25
98693 Ilmenau
Tel.: +49 3677 69 3621
Fax: +49 3677 69 3270
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Graph Edge Coloring: Tashkinov Trees and Goldberg's Conjecture

Diego Scheide

Institute of Mathematics

Technische Universität Ilmenau

D-98684 Ilmenau, Germany

Abstract

For the chromatic index $\chi'(G)$ of a (multi)graph G, there are two trivial lower bounds, namely the maximum degree $\Delta(G)$ and the density $\mathcal{W}(G) = \max_{H \subseteq G, |V(H)| \geq 2} \lceil |E(H)| / \lfloor |V(H)|/2 \rfloor \rceil$.

A famous conjectures due to Goldberg [3] and Seymour [10] says that every graph G satisfies $\chi'(G) \leq \max\{\Delta(G)+1, \mathcal{W}(G)\}$. This means that $\chi'(G) = \mathcal{W}(G)$ for every graph G with $\chi'(G) \geq \Delta(G) + 2$. The considered class of graphs $\mathcal{J}$ can be subdivided into an ascending sequence of classes $(\mathcal{J}_m)_{m \geq 3}$, and for $m \leq 13$ the conjecture is already proved. The "last" step was done by Favrholdt, Stiebitz and Toft [2] 2006, using and extending results from Tashkinov [12]. These methods are based on a coloring structure called Tashkinov tree. In this paper the same methods are used and extended to handle the "next" step $m \leq 15$. This leads to the result $\chi'(G) \leq \max\left\{\left\lfloor \frac{15}{14}\Delta(G) + \frac{12}{14} \right\rfloor, \mathcal{W}(G)\right\}$ for every graph G.

Furthermore, the used methods also lead to several improvements of other known upper bounds for the chromatic index. In particular, an asymptotic approximation of the chromatic index can be obtained. We prove that for every $\epsilon > 0$ and every graph G satisfying $\Delta(G) \geq \frac{1}{2\epsilon^2}$ the estimate $\chi'(G) \leq \max\{(1+\epsilon)\Delta(G), \mathcal{W}(G)\}$ holds. This extends a result of Kahn [5] as well as a result of Sanders and Steurer [7].

1 Notation

1.1 Graphs

By a *graph* we mean a finite undirected graph without loops, but possibly with multiple edges. The *vertex set* and the *edge set* of a graph G are denoted by $V(G)$

and $E(G)$ respectively. For a vertex $x \in V(G)$ let $E_G(x)$ denote the set of all edges of G that are incident with x. Two distinct edges of G incident to the same vertex will be called *adjacent edges*. Furthermore, if $X, Y \subseteq V(G)$ then let $E_G(X, Y)$ denote the set of all edges of G joining a vertex of X with a vertex of Y. We write $E_G(x, y)$ instead of $E_G(\{x\}, \{y\})$. Two distinct vertices $x, y \in V(G)$ with $E_G(x, y) \neq \emptyset$ will be called *adjacent vertices* or *neighbours*.

The *degree* of a vertex $x \in V(G)$ is $d_G(x) = |E_G(x)|$, and the *multiplicity* of two distinct vertices $x, y \in V(G)$ is $\mu_G(x, y) = |E_G(x, y)|$. Let $\Delta(G)$ and $\mu(G)$ denote the *maximum degree* and the *maximum multiplicity* of G respectively. A graph G is called *simple* if $\mu(G) \leq 1$.

For a graph G and a set $X \subseteq V(G)$ let $G[X]$ denote the subgraph *induced* by X, that is $V(G[X]) = X$ and $E(G[X]) = E_G(X, X)$. Further, let $G - X = G[V(G) \setminus X]$. We also write $G - x$ instead of $G - \{x\}$. For $F \subseteq E(G)$ let $G - F$ denote the subgraph H of G satisfying $V(H) = V(G)$ and $E(H) = E(G) \setminus F$. If $F = \{e\}$ is a singleton, we write $G - e$ rather than $G - \{e\}$.

By a path, a cycle or a tree we usually mean a graph or subgraph rather than a sequence consisting of edges and vertices. The only exception in this paper will be the Tashkinov tree. If P is a path of length p with $V(P) = \{v_0, \dots, v_p\}$ and $E(P) = \{e_1, \dots, e_p\}$ such that $e_i \in E_P(v_{i-1}, v_i)$ for $i = 1, \dots, p$, then we also write $P = P(v_0, e_1, v_1, \dots, e_p, v_p)$. Clearly, the vertices $v_0, \dots, v_p$ are distinct, and we say that v_0 and v_p are the *endvertices* of P or that P is a path *joining* v_0 and v_p.

1.2 Edge Colourings

By a *k-edge-coloring* of a graph G we mean a map $\varphi : E(G) \to \{1, \dots, k\}$ that assigns to every edge e of G a colour $\varphi(e) \in \{1, \dots, k\}$ such that no two adjacent edges receive the same colour. The set of all k-edge-colourings of G is denoted by $\mathcal{C}_k(G)$. The *chromatic index* or *edge chromatic number* $\chi'(G)$ is the smallest integer $k \geq 0$ such that $\mathcal{C}_k(G) \neq \emptyset$.

In the classic papers by Shannon [11] and Vizing [13, 14] a simple but very useful recolouring technique was developed, dealing with edge coloring problems in graphs. Suppose that G is a graph and φ is a k-edge-coloring of G. To obtain a new coloring, choose two distinct colours α, β, and consider the subgraph H with $V(H) = V(G)$ and $E(H) = \{e \in E(G) \,|\, \varphi(e) \in \{\alpha, \beta\}\}$. Then every component of H is either a path or an even cycle and we refer to such a component as an (α, β)*-chain* of G with respect to φ. Now choose an arbitrary (α, β)-chain C of G with respect to φ. If we change the colours α and β on C, then we obtain a k-edge-coloring φ' of G satisfying

$$\varphi'(e) = \begin{cases} \varphi(e) & \text{if } e \in E(G) \setminus E(C) \\ \beta & \text{if } e \in E(C) \text{ and } \varphi(e) = \alpha \\ \alpha & \text{if } e \in E(C) \text{ and } \varphi(e) = \beta. \end{cases}$$

In what follows, we briefly say that the coloring φ' is obtained from φ by recolouring C and write $\varphi' = \varphi/C$. Furthermore, for every vertex $v \in V(G)$ we denote by $P_v(\alpha, \beta, \varphi)$ the unique (α, β)-chain with respect to φ that contains the vertex v. For two vertices $v, w \in V(G)$, the chains $P_v(\alpha, \beta, \varphi)$ and $P_w(\alpha, \beta, \varphi)$ are either equal or vertex disjoint.

Consider a graph G and a coloring $\varphi \in \mathcal{C}_k(G)$. For a vertex $v \in V(G)$ we define the two colour sets

$$\varphi(v) = \{\varphi(e) \mid e \in E_G(v)\}$$

and

$$\bar{\varphi}(v) = \{1, \dots, k\} \setminus \varphi(v).$$

We call $\varphi(v)$ the set of colours *present* at v and $\bar{\varphi}(v)$ the set of colours *missing* at v with respect to φ. For a vertex set $X \subseteq V(G)$ we define $\bar{\varphi}(X) = \bigcup_{x \in X} \bar{\varphi}(x)$. If $\alpha, \beta \in \{1, \dots, k\}$ are two distinct colours and u, v are two distinct vertices of G satisfying $\alpha \in \bar{\varphi}(u)$ and $\beta \in \bar{\varphi}(v)$, then (u, v) is called an *(α, β)-pair* with respect to φ.

Let $\alpha, \beta \in \{1, \dots, k\}$ be two distinct colours. Moreover, let $v \in V(G)$ and $P = P_v(\alpha, \beta, \varphi)$. If v is a vertex v of G such that exactly one of the two colours α or β is missing at v with respect to φ, then P is a path where one endvertex is v and the other endvertex is some vertex $u \neq v$ such that either α or β is missing at u. For the coloring $\varphi' = \varphi/P$ we have $\varphi' \in \mathcal{C}_k(G)$. Moreover, if w is an endvertex of P then we have

$$\bar{\varphi}'(w) = \begin{cases} (\bar{\varphi}(w) \setminus \{\beta\}) \cup \{\alpha\} & \text{if } \bar{\varphi}(w) \cap \{\alpha, \beta\} = \{\beta\} \\ (\bar{\varphi}(w) \setminus \{\alpha\}) \cup \{\beta\} & \text{if } \bar{\varphi}(w) \cap \{\alpha, \beta\} = \{\alpha\} \\ (\bar{\varphi}(w) & \text{if } \bar{\varphi}(w) \cap \{\alpha, \beta\} = \{\alpha, \beta\} \end{cases}$$

For all other vertices w beside the endvertices of P, we have $\bar{\varphi}'(w) = \bar{\varphi}(w)$. These facts shall be used quite often without mentioning it explicitly.

1.3 Critical graphs

Let G be a graph. We call G *critical* (with respect to χ') if $\chi'(H) < \chi'(G)$ for every proper subgraph H of G. If G is critical and $\chi'(G) = k$ then we also say that G is *k-critical.* We call $e \in E(G)$ a *critical edge* of G if $\chi'(G - e) = \chi'(G) - 1$. Clearly, G is critical if and only if G is connected and every edge of G is critical. Moreover, $\chi'(G) \leq k$ if and only if G does not contain a $(k + 1)$-critical subgraph.

2 Elementary graphs

Consider a graph G and a coloring $\varphi \in \mathcal{C}_k(G)$. Clearly for every colour $\gamma \in \{1, \dots, k\}$ and every subgraph H of G with $|V(H)| \geq 2$ the edge set $E_\gamma(H) = \{e \in$

$E(H) \mid \varphi(e) = \gamma\}$ is a matching in H. Consequently we have $|E_\gamma(H)| \leq \frac{|V(H)|}{2}$, and therefore $|E(H)| \leq k \left\lfloor \frac{|V(H)|}{2} \right\rfloor$. This observation leads to the following parameter for a graph G with $|V(G)| \geq 2$, namely the *density*

$$\mathcal{w}(G) = \max_{\substack{H \subseteq G \\ |V(H)| \geq 2}} \left\lceil \frac{|E(H)|}{\lfloor \frac{1}{2}|V(H)| \rfloor} \right\rceil.$$

For a graph G with $|V(G)| \leq 1$ define $\mathcal{w}(G) = 0$. Then, clearly, $\chi'(G) \geq \mathcal{w}(G)$ for every graph G. A graph G satisfying $\chi'(G) = \mathcal{w}(G)$ is called an *elementary graph.* The following conjecture seems to have been thought of first by Goldberg [3] around 1970 and, independently, by Seymour [10] in 1977.

Conjecture 2.1 (Goldberg [3] 1973 and Seymour [10] 1979) *Every graph G with $\chi'(G) \geq \Delta(G) + 2$ is elementary.*

The parameter $\mathcal{w}$ is related to the so-called fractional chromatic index. A *fractional edge coloring* of a graph G is an assignment of a non-negative weight w_M to each matching M of G such that for every edge $e \in E(G)$ we have

$$\sum_{M : e \in M} w_M \geq 1.$$

Then the *fractional chromatic index* $\chi'_f(G)$ is the minimum value of $\sum_M w_M$ where the sum is over all matchings M of G and the minimum is over all fractional edge colourings w of G. In case of $|E(G)| = 0$ we have $\chi'_f(G) = 0$. From the definition follows that $\chi'_f(G) \leq \chi'(G)$ for every graph G. The computation of the chromatic index is NP-hard, but with matching techniques one can compute the fractional chromatic index in polynomial time, see [8, 9] for a proof.

From Edmonds' matching polytope theorem the following characterization of the fractional chromatic index of an arbitrary graph G can be obtained (see [8, 9] for details):

$$\chi'_f(G) = \max\{\Delta(G), \max_{\substack{H \subseteq G \\ |V(H)| \geq 2}} \frac{|E(H)|}{\lfloor \frac{1}{2}|V(H)| \rfloor}\}.$$

As an immediate consequence of this characterization we obtain $\mathcal{w}(G) \leq \Delta(G)$ if $\chi'_f(G) = \Delta(G)$, and $\mathcal{w}(G) = \lceil \chi'_f(G) \rceil$ if $\chi'_f(G) > \Delta(G)$. This implies, that Goldberg's conjecture is equivalent to the claim that $\chi'(G) = \lceil \chi'_f(G) \rceil$ for every graph G with $\chi'(G) \geq \Delta(G) + 2$.

The following result due to Kahn [5] shows that the fractional chromatic index asymptotically approximates the chromatic index.

Theorem 2.2 (Kahn [5] 1995) *For every $\epsilon \geq 0$ there is a Δ_ϵ such that for any graph G with $\chi'_f(G) > \Delta_\epsilon$ we have $\chi'(G) < (1 + \epsilon)\chi'_f(G)$.*

This result was proven by probabilistic methods. We will extend this result and show, using constructive colouring arguments, that for every $\epsilon > 0$ every graph G with $\Delta(G) \geq \frac{1}{2\epsilon^2}$ satisfies $\chi'(G) \leq \max\{(1+\epsilon)\Delta(G), w(G)\}$, see Section 5.

Another equivalent formulation of Goldberg's conjecture can be obtained as follows. For an integer $m \geq 3$ let $\mathcal{J}_m$ denote the class of all graphs G with

$$\chi'(G) > \frac{m}{m-1}\Delta(G) + \frac{m-3}{m-1}.$$

Then for every integer $m \geq 3$ we have $\mathcal{J}_m \subseteq \mathcal{J}_{m+1}$. Moreover, the class

$$\mathcal{J} = \bigcup_{m=3}^{\infty} \mathcal{J}_m$$

consists of all graphs G with $\chi'(G) \geq \Delta(G)+2$. Consequently, there is an equivalent formulation for Goldberg's Conjecture:

Conjecture 2.3 *Let $m \geq 3$ be an integer. Then every graph G with $\chi'(G) > \frac{m}{m-1}\Delta(G) + \frac{m-3}{m-1}$ is elementary.*

Up to now, this conjecture is known to be true for $m \leq 13$. It was proved for $m = 5$ by Sørensen (unpublished), Andersen [1] and Goldberg [3], for $m = 7$ by Sørensen (unpublished) and Andersen [1], for $m = 9$ by Goldberg [4], for $m = 11$ by Nishizeki and Kashiwagi [6] and by Tashkinov [12] and, eventually, for $m = 13$ by Favrholdt, Stiebitz and Toft [2]. The methods used by Tashkinov [12] and extended by Favrholdt, Stiebitz and Toft [2], shall be used here, too, to show the following result.

Theorem 2.4 *Every graph G with*

$$\chi'(G) > \frac{15}{14}\Delta(G) + \frac{12}{14}$$

is elementary.

As an immediate consequence of Theorem 2.4 we get the following upper bound for the chromatic index.

Corollary 2.5 *Every graph G satisfies*

$$\chi'(G) \leq \max\left\{\left\lfloor \frac{15}{14}\Delta(G) + \frac{12}{14} \right\rfloor, w(G)\right\}.$$

In order to prove Theorem 2.4 it is sufficient to show that every critical graph in $\mathcal{J}_{15}$ is elementary. This follows from the next result.

Proposition 2.6 *Let $m \geq 3$ be an odd integer. If every critical graph in $\mathcal{J}_m$ is elementary, then every graph in $\mathcal{J}_m$ is elementary, too.*

Proof: Let $G \in \mathcal{J}_m$. Clearly, G contains a critical subgraph H with $\chi'(H) = \chi'(G)$. Since $\Delta(H) \leq \Delta(G)$, we have $H \in \mathcal{J}_m$. Then, by assumption, H is an elementary graph. This implies $w(G) \leq \chi'(G) = \chi'(H) = w(H) \leq w(G)$ and therefore $\chi'(G) = w(G)$. Hence G is an elementary graph, too. ■

The concept of elementary graphs is closely related to the concept of elementary sets. Consider a graph G, an edge $e \in E(G)$, a coloring $\varphi \in \mathcal{C}_k(G-e)$ and a vertex set $X \subseteq V(G)$. Then X is called *elementary* with respect to φ if $\bar{\varphi}(u) \cap \bar{\varphi}(v) = \emptyset$ for every two distinct vertices $u, v \in X$. The following result provides some basic facts about elementary sets which will be useful for our further investigations.

Proposition 2.7 (Favrholdt, Stiebitz and Toft [2] 2006) *Let G be a graph with $\chi'(G) = k+1$ for an integer $k \geq \Delta = \Delta(G)$, and let $e \in E(G)$ be a critical edge of G. If $X \subseteq V(G)$ is an elementary set with respect to a coloring $\varphi \in \mathcal{C}_k(G-e)$ such that both endvertices of e are contained in X, then the following statements hold:*

(a) $|X| \leq \frac{|\bar{\varphi}(X)|-2}{k-\Delta} \leq \frac{k-2}{k-\Delta}$ *provided that $k \geq \Delta + 1$*

(b) $\sum_{v \in X} d_G(v) \geq k(|X|-1)+2$

(c) *Suppose that*
$$\chi'(G) > \frac{m}{m-1}\Delta(G) + \frac{m-3}{m-1}$$
for an integer $m \geq 3$. Then $|X| \leq m-1$ and, moreover, $|\bar{\varphi}(X)| \geq \Delta + 1$ provided that $|X| = m-1$.

Let G be a graph and let $e \in E(G)$. Moreover, let $X \subseteq V(G)$ and $\varphi \in \mathcal{C}_k(G-e)$. The set X is called *closed* with respect to φ if for every edge $f \in E_G(X, V(G) \setminus X)$ the colour $\varphi(f)$ is present at every vertex of X, i.e. $\varphi(f) \in \varphi(v)$ for every $v \in X$. Furthermore, the set X is called *strongly closed* with respect to φ if X is closed and $\varphi(f) \neq \varphi(f')$ for every two distinct edges $f, f' \in E_G(X, V(G) \setminus X)$. The relations between elementary graphs and elementary and strongly closed sets are shown by the following result, which is implicitly contained in the papers by Andersen [1] and Goldberg [4]. A proof of this theorem can be found in [2].

Theorem 2.8 (Favrholdt, Stiebitz and Toft [2] 2006) *Let G be a graph with $\chi'(G) = k+1$ for an integer $k \geq \Delta(G)$. If G is critical, then the following conditions are equivalent.*

(a) *G is elementary.*

(b) *For every edge $e \in E(G)$ and every coloring $\varphi \in \mathcal{C}_k(G-e)$ the set $V(G)$ is elementary with respect to φ.*

(c) *There is an edge $e \in E(G)$ and a coloring $\varphi \in \mathcal{C}_k(G-e)$ such that $V(G)$ is elementary with respect to φ.*

(d) *There is an edge $e \in E(G)$, a coloring $\varphi \in \mathcal{C}_k(G-e)$ and a set $X \subseteq V(G)$ such that X contains the two endvertices of e and X is elementary as well as strongly closed with respect to φ.*

This gives the approach to prove Theorem 2.4. We will show, that for every critical graph in $\mathcal{J}_{15}$ there is an edge $e \in E_G(x,y)$, a coloring $\varphi \in \mathcal{C}_k(G-e)$ and a vertex set $X \subseteq V(G)$ such that $x,y \in X$ and X is both elementary and strongly closed with respect to φ. This, by Theorem 2.8 and Proposition 2.6, will be sufficient to prove Theorem 2.4. The construction of the desired vertex set X highly differs, depending on several properties of the graph G. These cases are independently analysed in Section 4. Due to theses results, the proof itself of Theorem 2.4 is presented at the end of Section 4. In preparation the necessary methods and results from Tashkinov [12] and Favrholdt, Stiebitz and Toft [2] will be summarized and extended in the next section.

3 Tashkinov trees

3.1 The basic Tashkinov tree

Definition 3.1 *Let G be a graph and let $e \in E(G)$ be an edge such that $\chi'(G-e) = k$. By a* Tashkinov tree *with respect to e and a coloring $\varphi \in \mathcal{C}_k(G-e)$ we mean a sequence $(y_0, e_1, y_1, \ldots, e_n, y_n)$ consisting of edges $e_1, \ldots, e_n$ and vertices $y_0, \ldots, y_n$ satisfying the following two conditions:*

(T1) *The vertices $y_0, \ldots, y_n$ are distinct, $e_1 = e$ and for $i = 1, \ldots, n$ we have $e_i \in E_G(y_i, y_j)$ where $0 \leq j < i$.*

(T2) *For every edge e_i with $2 \leq i \leq n$ there is a vertex y_j with $0 \leq j < i$ such that $\varphi(e_i) \in \bar{\varphi}(y_j)$.*

For a Tashkinov tree $T = (y_0, e_1, y_1, \ldots, e_n, y_n)$ with respect to e and φ we intuitively denote $V(T) = \{y_0, \ldots, y_n\}$ and $E(T) = \{e_1, \ldots, e_n\}$. In this paper Tashkinov trees will be the basic structures for coloring graphs. One of their most important properties is their elementarity as the following result states.

Theorem 3.2 (Tashkinov [12] 2000) *Let G be a graph with $\chi'(G) = k+1$ for an integer $k \geq \Delta(G)+1$, let $e \in E(G)$ be a critical edge of G and let $\varphi \in \mathcal{C}_k(G-e)$. If T is a Tashkinov tree with respect to e and φ, then $V(T)$ is elementary with respect to φ.*

Let $T = (y_0, e_1, y_1, \ldots, e_n, y_n)$ be a Tashkinov tree with respect to e and φ. Clearly $Ty_r = (y_0, e_1, y_1, \ldots, e_r, y_r)$, where $1 \leq r \leq n$, is a Tashkinov tree with respect to e and φ, too.

We say that a colour α is *used* on T with respect to φ if $\varphi(f) = \alpha$ for some edge $f \in E(T)$. Otherwise we say that α is *unused* on T with respect to φ.

Theorem 3.3 (Favrholdt, Stiebitz and Toft [2] 2006) *Let G be a graph with $\chi'(G) = k+1$ for an integer $k \geq \Delta(G)+1$, let $e \in E(G)$ be a critical edge of G and let $\varphi \in \mathcal{C}_k(G-e)$ be a coloring. Moreover, let T be a maximal Tashkinov tree with respect to e and φ and let $T' = (y_0, e_1, y_1, \ldots, e_n, y_n)$ be an arbitrary Tashkinov tree with respect to e and φ. Then the following statements hold:*

(a) *$V(T)$ is elementary and closed both with respect to φ.*

(b) *$|V(T)|$ is odd.*

(c) *$V(T') \subseteq V(T)$.*

(d) *There is a Tashkinov tree $\tilde{T}$ with respect to e and φ satisfying $V(\tilde{T}) = V(T)$ and $\tilde{T}y_n = T'$.*

(e) *Suppose that (y_i, y_j) is a (γ, δ)-pair with respect to φ where $0 \leq i < j \leq n$. Then $\gamma \neq \delta$ and there is a (γ, δ)-chain P with respect to φ satisfying the following conditions:*

  (1) *P is a path with endvertices y_i and y_j.*

  (2) *$|E(P)|$ is even.*

  (3) *$V(P) \subseteq V(T)$.*

  (4) *If γ is unused on $T'y_j$ with respect to φ, then T' is a Tashkinov tree with respect to the edge e and the coloring $\varphi' \in \mathcal{C}_k(G-e)$ obtained from φ by recolouring the (γ, δ)-chain P.*

(f) *$G[V(T)]$ contains an odd cycle.*

3.2 Extended Tashkinov trees

Let G be a critical graph with $\chi'(G) = k+1$ for a given integer $k \geq \Delta(G)+1$. Since G is critical, for every edge $e \in E(G)$ and every coloring $\varphi \in \mathcal{C}_k(G-e)$ there is a Tashkinov tree T with respect to e and φ. Hence there is a largest number n such that $n = |V(T)|$ for such a Tashkinov tree T. We call n the *Tashkinov order* of G and write $t(G) = n$. Furthermore, we denote by $\mathcal{T}(G)$ the set of all triples (T, e, φ) such that $e \in E(G)$, $\varphi \in \mathcal{C}_k(G-e)$ and T is a Tashkinov tree on n vertices with respect to e and φ. Evidently $\mathcal{T}(G) \neq \emptyset$.

For a triple $(T, e, \varphi) \in \mathcal{T}(G)$ we introduce the following notations. For a colour α let $E_\alpha(e, \varphi) = \{e' \in E(G) \backslash \{e\} \mid \varphi(e') = \alpha\}$ the set of all edges of G coloured with α with respect to φ. Further let

$$E_\alpha(T, e, \varphi) = E_\alpha(e, \varphi) \cap E_G(V(T), V(G) \backslash V(T)).$$

The colour α is said to be *defective* with respect to (T, e, φ) if $|E_\alpha(T, e, \varphi)| \geq 2$. The set of all defective colours with respect to (T, e, φ) is denoted by $\Gamma^d(T, e, \varphi)$. The colour α is said to be *free* with respect to (T, e, φ) if $\alpha \in \bar{\varphi}(V(T))$ and α is unused on T with respect to φ. The set of all free colours with respect to (T, e, φ) is denoted by $\Gamma^f(T, e, \varphi)$.

Proposition 3.4 (Favrholdt, Stiebitz and Toft [2] 2006) *Let G be a critical graph with $\chi'(G) = k+1$ for an integer $k \geq \Delta(G)+1$, and let $(T, e, \varphi) \in \mathcal{T}(G)$. Then the following statements hold:*

(a) $|V(T)| = t(G) \geq 3$ *is odd.*

(b) $V(T)$ *is elementary and closed both with respect to* φ.

(c) $V(T)$ *is strongly closed with respect to* φ *if* $\Gamma^d(T, e, \varphi) = \emptyset$.

(d) *If* $\alpha \in \bar{\varphi}(V(T))$ *then* $E_\alpha(T, e, \varphi) = \emptyset$.

(e) *If* $\alpha \in \Gamma^d(T, e, \varphi)$ *then* $|E_\alpha(T, e, \varphi)| \geq 3$ *is odd.*

(f) *For a vertex* $v \in V(T)$ *we have*

$$|\bar{\varphi}(v)| = \begin{cases} k - d_G(v) + 1 & \geq 2 \quad \text{if } e \in E_G(v) \\ k - d_G(v) & \geq 1 \quad \text{otherwise.} \end{cases}$$

(g) $|\Gamma^f(T, e, \varphi)| \geq 4$.

(h) *Every colour in* $\Gamma^d(T, e, \varphi) \cup \Gamma^f(T, e, \varphi)$ *is unused on* T *with respect to* φ.

Let G be a critical graph with $\chi'(G) = k+1$ where $k \geq \Delta(G)+1$, and let $(T, e, \varphi) \in \mathcal{T}(G)$. A vertex v in $V(G) \setminus V(T)$ is called *absorbing* with respect to (T, e, φ), if for every colour $\delta \in \bar{\varphi}(v)$ and every free colour $\gamma \in \Gamma^f(T, e, \varphi)$ with $\gamma \neq \delta$ the (γ, δ)-chain $P_v(\gamma, \delta)$ contains a vertex $u \in V(T)$ satisfying $\gamma \in \bar{\varphi}(u)$. Since, by Proposition 3.4(b), $V(T)$ is elementary with respect to φ, this vertex u is the unique vertex in T with $\gamma \in \bar{\varphi}(u)$ and, moreover, $P_v(\gamma, \delta)$ is a path whose endvertices are u and v. Clearly u belongs to $P_v(\gamma, \delta)$ if and only if v belongs to $P_u(\gamma, \delta)$. Let $A(T, e, \varphi)$ denote the set of all vertices in $V(G) \setminus V(T)$ which are absorbing with respect to (T, e, φ).

Proposition 3.5 (Favrholdt, Stiebitz and Toft [2] 2006) *Let G be a critical graph with $\chi'(G) = k+1$ for an integer $k \geq \Delta(G)+1$, and let $(T, e, \varphi) \in \mathcal{T}(G)$. Then the vertex set $V(T) \cup A(T, e, \varphi)$ is elementary with respect to φ.*

Let P be a path and let u, v be two vertices of P. Then there is a unique subpath P' of P having u and v as endvertices. We denote this subpath by uPv or vPu. If we fix an endvertex of P, say w, then we obtain a linear order $\preceq_{(w,P)}$ of the vertex set of P in a natural way, where $x \preceq_{(w,P)} y$ if the vertex x belongs to the subpath wPy.

Proposition 3.6 (Favrholdt, Stiebitz and Toft [2] 2006) *Let G be a critical graph with $\chi'(G) = k+1$ for an integer $k \geq \Delta(G)+1$, and let $(T, e, \varphi) \in \mathcal{T}(G)$. Let $\alpha \in \Gamma^d(T, e, \varphi)$ be a defective colour and let u be a vertex of T such that $\bar{\varphi}(u)$ contains a free colour $\gamma \in \Gamma^f(T, e, \varphi)$. Then for the (α, γ)-chain $P = P_u(\alpha, \gamma, \varphi)$ the following statements hold:*

(a) *P is a path where one endvertex is u and the other endvertex is some vertex $v \in V(G) \setminus V(T)$.*

(b) $E_\alpha(T, e, \varphi) = E(P) \cap E_G(V(T), V(G) \setminus V(T))$.

(c) *In the linear order $\preceq_{(u,P)}$ there is a first vertex v^1 that belongs to $V(G) \setminus V(T)$ and there is a last vertex v^2 that belongs to $V(T)$ where $v^1 \preceq_{(u,P)} v^2$.*

(d) $\bar{\varphi}(v^2) \cap \Gamma^f(T, e, \varphi) = \emptyset$.

(e) *$V(T) \cup \{v^1\}$ is elementary with respect to φ.*

We call $v \in V(G)$ a *defective vertex* with respect to (T, e, φ) if there are two distinct colours α and γ such that $\alpha \in \Gamma^d(T, e, \varphi)$ is a defective colour, $\gamma \in \Gamma^f(T, e, \varphi)$ is a free colour and v is the first vertex in the linear order $\preceq_{(u,P)}$ that belongs to $V(G) \setminus V(T)$, where u is the unique vertex in T with $\gamma \in \bar{\varphi}(u)$ and $P = P_u(\alpha, \gamma)$. The set of all defective vertices with respect to (T, e, φ) is denoted by $D(T, e, \varphi)$.

Proposition 3.7 (Favrholdt, Stiebitz and Toft [2] 2006) *Let G be a critical graph with $\chi'(G) = k+1$ for an integer $k \geq \Delta(G)+1$, and let $(T,e,\varphi) \in \mathcal{T}(G)$. Then $D(T,e,\varphi) \subseteq A(T,e,\varphi)$.*

Consider an arbitrary triple $(T,e,\varphi) \in \mathcal{T}(G)$. Let $\gamma \in \bar{\varphi}(u)$ for a vertex $u \in V(T)$ and let $\delta \in \Gamma^d(T,e,\varphi)$. Clearly, the (γ,δ)-chain $P = P_u(\gamma,\delta,\varphi)$ is a path where u is one endvertex of P and, moreover, exactly one of the two colours γ or δ is missing at the second endvertex of P with respect to φ. Since $V(T)$ is elementary and δ is present at every vertex in $V(T)$, both with respect to φ, the second endvertex of P belongs to $V(G)\backslash V(T)$. Hence in the linear order $\preceq_{(u,P)}$ there is a last vertex v that belongs to $V(T)$. This vertex is said to be an *exit vertex* with respect to (T,e,φ). The set of all exit vertices with respect to (T,e,φ) is be denoted by $F(T,e,\varphi)$.

Lemma 3.8 *Let G be a critical graph with $\chi'(G) = k+1$ for an integer $k \geq \Delta(G)+1$, and let $(T,e,\varphi) \in \mathcal{T}(G)$. Then $\bar{\varphi}(F(T,e,\varphi)) \cap \Gamma^f(T,e,\varphi) = \emptyset$.*

Proof: Let $v \in F(T,e,\varphi)$. Then there is a vertex $u \in V(T)$, a colour $\gamma \in \bar{\varphi}(u)$ and a colour $\delta \in \Gamma^d(T,e,\varphi)$ such that v is the last vertex in the linear order $\preceq_{(u,P)}$ that belongs to $V(T)$, where $P = P_u(\gamma,\delta,\varphi)$. Clearly, P is a path with one endvertex u and another endvertex $z \in V(G) \setminus V(T)$.

Suppose there is a colour $\alpha \in \bar{\varphi}(v) \cap \Gamma^f(T,e,\varphi)$. By Proposition 3.4(b), $V(T)$ is elementary and closed with respect to φ, and therefore no edge in $E_G(V(T), V(G) \setminus V(T))$ is coloured with α or γ with respect to φ. Hence there is a coloring $\varphi' \in \mathcal{C}_k(G-e)$, obtained from φ by exchanging the colours α and γ on all edges in $E_G(V(G) \setminus V(T), V(G) \setminus V(T))$. Then evidently $(T,e,\varphi') \in \mathcal{T}(G)$, $\Gamma^f(T,e,\varphi') = \Gamma^f(T,e,\varphi)$ and $\Gamma^d(T,e,\varphi') = \Gamma^d(T,e,\varphi)$. In particular, we have $\alpha \in \bar{\varphi}'(v) \cap \Gamma^f(T,e,\varphi')$ and $\delta \in \Gamma^d(T,e,\varphi')$. Moreover, for $P' = P_v(\alpha,\delta,\varphi')$ we have $P' = vPz$ and, therefore, on the one hand we have $|E(P') \cap E_G(V(T), V(G) \setminus V(T)| = 1$. On the other hand Proposition 3.6(b) implies $|E(P') \cap E_G(V(T), V(G) \setminus V(T)| = |E_\delta(T,e,\varphi')| > 1$, a contradiction. Hence we have $\bar{\varphi}(v) \cap \Gamma^f(T,e,\varphi) = \emptyset$ and the proof is completed. ■

Definition 3.9 *Let G be a critical graph with $\chi'(G) = k+1$ where $k \geq \Delta(G)+1$, and let $(T,e,\varphi) \in \mathcal{T}(G)$. Furthermore let Z be a vertex set with $V(T) \subseteq Z \subseteq V(G)$. A sequence $F = (e_1, u_1, \ldots, e_p, u_p)$ is called a* fan *at Z with respect to φ if the following conditions hold:*

(F1) *The edges $e_1, \ldots, e_p \in E(G)$ as well as the vertices $u_1, \ldots, u_p \in V(G)$ are distinct.*

(F2) *For every $i \in \{1, \ldots, p\}$ there are two vertices $z \in Z$ and $z' \in Z \cup \{u_1, \ldots, u_{i-1}\}$ satisfying $e_i \in E_G(z, u_i)$ and $\varphi(e_i) \in \bar{\varphi}(z')$.*

Theorem 3.10 (Favrholdt, Stiebitz and Toft [2] 2006) *Let G be a critical graph with $\chi'(G) = k + 1$ for an integer $k \geq \Delta(G) + 1$, and let $(T, e, \varphi) \in \mathcal{T}(G)$. Furthermore let $Y \subseteq D(T, e, \varphi)$ and $Z = V(T) \cup Y$. If F is a fan at Z with respect to φ, then $Z \cup V(F)$ is elementary with respect to φ.*

3.3 Normalized Tashkinov trees

Consider an arbitrary triple $(T, e, \varphi) \in \mathcal{T}(G)$. Then T has the form

$$T = (y_0, e_1, y_1, \ldots, e_{n-1}, y_{n-1})$$

where $n = t(G)$. Then T is called a *normal Tashkinov tree* with respect to e and φ if there are two colours $\alpha \in \bar{\varphi}(y_0)$ and $\beta \in \bar{\varphi}(y_1)$, an integer $2 \leq p \leq n-1$ and an edge $f \in E_G(y_0, y_{p-1})$ such that $P(y_1, e_2, y_2, \ldots, e_{p-1}, y_{p-1}, f, y_0)$ is an (α, β)-chain with respect to φ. In this case Ty_{p-1} is called the (α, β)-*trunk* of T and the number p is called the *height* of T denoted by $h(T) = p$. Furthermore let $\mathcal{T}^N(G)$ denote the set of all triples $(T, e, \varphi) \in \mathcal{T}(G)$ for which T is a normal Tashkinov tree, and let $h(G)$ denote the greatest number p such that there is a triple $(T, e, \varphi) \in \mathcal{T}^N(G)$ with $h(T) = p$. The following lemma shows that normal Tashkinov trees can be generated from arbitrary ones, which also implies that $\mathcal{T}^N(G) \neq \emptyset$.

Lemma 3.11 *Let G be a critical graph with $\chi'(G) = k+1$ for an integer $k \geq \Delta(G)+1$, and let $(T, e, \varphi) \in \mathcal{T}(G)$ with $e \in E_G(x, y)$. Then there are two colours $\alpha \in \bar{\varphi}(x)$ and $\beta \in \bar{\varphi}(y)$, and there is a Tashkinov tree T' with respect to e and φ satisfying $V(T') = V(T)$, $(T', e, \varphi) \in \mathcal{T}^N(G)$ and $h(T') = |V(P)|$ where $P = P_x(\alpha, \beta, \varphi)$.*

Proof: Since $k \geq \Delta(G) + 1$ there are two colours $\alpha \in \bar{\varphi}(x)$ and $\beta \in \bar{\varphi}(y)$. By Theorem 3.3(e) we have $\alpha \neq \beta$, and P is an (α, β)-chain with respect to φ having endvertices x and y. This means P is a path of the form

$$P = P(v_1, f_2, v_2, \ldots, f_p, v_p)$$

with $v_1 = y$ and $v_p = x$. Evidently $f_j \in E_G(v_{j-1}, v_j)$ and $\varphi(f_j) \in \{\alpha, \beta\} \subseteq \bar{\varphi}(\{x, y\})$ for all $j \in \{2, \ldots, p\}$. Hence $T_1 = (x, e, y, f_2, v_2, \ldots, f_{p-1}, v_{p-1})$ is a Tashkinov tree with respect to e and φ. By Theorem 3.3(d) there is a Tashkinov tree T' with respect to e and φ with $V(T') = V(T)$ and $T'v_{p-1} = T_1$. Then T' is normal with respect to e and φ, and therefore $(T', e, \varphi) \in \mathcal{T}^N(G)$. Moreover, T_1 is the (α, β)-trunk of T' and $h(T') = |V(T_1)| = |V(P)|$, which completes the proof. ■

Consider an arbitrary triple $(T, e, \varphi) \in \mathcal{T}^N(G)$ where T has the form $T = (y_0, e_1, y_1, \ldots, e_{n-1}, y_{n-1})$. Then (T, e, φ) is called a *balanced triple* with respect to e and φ if $h(T) = p = h(G)$ and $\varphi(e_{2j}) = \varphi(e_{2j-1})$ for $p < 2j < n$. Let $\mathcal{T}^B(G)$ denote the set of all balanced triples $(T, e, \varphi) \in \mathcal{T}(G)$. The following lemma shows that $\mathcal{T}^B(G) \neq \emptyset$.

Lemma 3.12 *Let G be a critical graph with $\chi'(G) = k+1$ for an integer $k \geq \Delta(G)+1$. Then the following statements hold:*

(a) *$h(G) \geq 3$ is odd.*

(b) *If $(T, e, \varphi) \in \mathcal{T}^N(G)$ with $h(T) = h(G)$, then there is a Tashkinov tree T' with respect to e and φ with $V(T') = V(T)$ and $(T', e, \varphi) \in \mathcal{T}^B(G)$, and moreover, all colours used on T' are used on T.*

(c) *$\mathcal{T}^B(G) \neq \emptyset$.*

Proof: Let $(T, e, \varphi) \in \mathcal{T}^N(G)$ with $h(T) = h(G) = p$. Then T has the form $T = (y_0, e_1, y_1, \ldots, e_{n-1}, y_{n-1})$ and Ty_p is the (α, β)-trunk of T. By definition we have $\alpha \in \bar{\varphi}(y_0)$, $\beta \in \bar{\varphi}(y_1)$ and there is an edge $f \in E_G(y_0, y_{p-1})$ satisfying $P = P_{y_0}(\alpha, \beta, \varphi) = P(y_1, \ldots, e_{p-1}, y_{p-1}, f, y_0)$. Hence P is a path with two distinct endvertices, and moreover P is alternately coloured with two colours with respect to φ. Since every of these two colours is missing at one of the endvertices, we conclude that $|E(P)| \geq 2$ is even. Hence $p = h(G) \geq 3$ is odd and (a) is proved.

Now let $i \leq n$ the greatest odd integer for which there exists a Tashkinov tree $T' = (y'_0, e'_1, y'_1, \ldots, e'_{i-1}, y'_{i-1})$ with respect to e and φ satisfying $T'y'_{p-1} = Ty_{p-1}$, $\varphi(E(T')) \subseteq \varphi(E(T))$ and $\varphi(e'_{2j-1}) = \varphi(e'_{2j})$ for $p < 2j < i$. Evidently, we have $i \geq p$, since T fulfils these requirements for $i = p$.

Now suppose that $i < n$. Then there is a smallest integer r satisfying $y_r \in V(T) \setminus V(T')$. Let $y'_i = y_r$ and $e'_i = e_r$. Hence $e'_i \in E_G(V(T'), y'_i)$ and $\varphi(e'_i) \in \bar{\varphi}(V(T'))$, and therefore $T_1 = (T', e'_i, y'_i)$ is a Tashkinov tree with respect to e and φ. Let $\gamma = \varphi(e'_i)$. Clearly $|V(T_1)|$ is even, $\gamma \in \bar{\varphi}(V(T_1))$ and, by Theorem 3.2, $V(T_1)$ is elementary with respect to φ. Hence there is an edge $e'_{i+1} \in E_G(V(T_1), y'_{i+1})$ satisfying $y'_{i+1} \in V(G) \setminus V(T_1)$ and $\varphi(e'_{i+1}) = \gamma$. Evidently $T_2 = (T_1, e'_{i+1}, y'_{i+1})$ is a Tashkinov tree with respect to e and φ satisfying $T_2y'_{p-1} = Ty_{p-1}$, $\varphi(E(T_2)) \subseteq \varphi(E(T))$ and $\varphi(e'_{2j-1}) = \varphi(e'_{2j})$ for $p < 2j < i+2$. This contradicts the maximality of i. Consequently, we have $i = n$ and, by Theorem 3.3(d), $V(T') = V(T)$. Hence $(T', e, \varphi) \in \mathcal{T}^B(G)$ with $\varphi(E(T')) \subseteq \varphi(E(T))$ and (b) is proved. Furthermore, (c) follows simply from (b) and the fact that $\mathcal{T}^N(G) \neq \emptyset$. This completes the proof. ■

Consider a graph G and a balanced triple $(T, e, \varphi) \in \mathcal{T}^B(G)$. Then T has the form

$$T = (y_0, e_1, y_1, \ldots, e_{n-1}, y_{n-1})$$

and Ty_p is the (α, β)-trunk of T, where $p = h(G)$, $\alpha \in \bar{\varphi}(y_0)$ and $\beta \in \bar{\varphi}(y_1)$. Moreover, there is an edge $f_p \in E_G(y_0, y_{p-1})$ with $\varphi(e_0) = \beta$. For $i = 1, \ldots, p-1$ let $f_i = e_i$. Clearly, the edges $f_1, \ldots, f_p$ form a cycle in G. Furthermore, the edge $f_1 = e$ is uncoloured and the edges $f_2, \ldots, f_p$ are coloured alternately with α and β with respect to φ. Now choose a $j \in \{1, \ldots, p-1\}$. Since (y_0, y_1) is a (α, β)-pair with respect to φ, there is a coloring $\varphi' \in \mathcal{C}_k(G - f_{j+1})$ such that $\varphi'(e') = \varphi(e')$ for all edges $e' \in E(G) \setminus \{f_1, \ldots, f_p\}$ and the edges $f_{j+2}, \ldots, f_p, f_1, \ldots, f_j$ are coloured alternately with α and β with respect to φ'. Then

$$T' = (y_j T y_{p-1}, f_p, y_0 T y_{j-1}, e_p, y_p, \ldots, e_{n-1}, y_{n-1})$$

is a normal Tashkinov tree where $T'y_{j-1}$ is the (α, β)-trunk of T' and, moreover, the triple (T', f_{j+1}, φ') is balanced. We denote the new triple by $(T', f_{j+1}, \varphi') = (T, e, \varphi)(y_0 \to y_j)$.

4 Tashkinov trees and elementary graphs

In this section we will develop several conditions related to Tashkinov trees, which will imply that a critical graph is elementary. Some of these results are generalizations of results implicitly given in [2], others, like the following, are new ones. The next lemma analyses Tashkinov trees with a special structure, they will be used to handle some of the cases for the proof of Theorem 2.4.

Lemma 4.1 *Let G be a critical graph with $\chi'(G) = k + 1$ for an integer $k \geq \Delta(G) + 1$, and let $(T, e, \varphi) \in \mathcal{T}(G)$. Furthermore let T be of the form*

$$T = (y_0, e_1, y_1, \ldots, e_{p-1}, y_{p-1}, f^1_{\gamma_1}, u^1_{\gamma_1}, f^2_{\gamma_1}, u^2_{\gamma_1}, \ldots, f^1_{\gamma_s}, u^1_{\gamma_s}, f^2_{\gamma_s}, u^2_{\gamma_s})$$

where $\Gamma = \{\gamma_1, \ldots, \gamma_s\}$ is a set of s colours, $Y = \{y_0, \ldots, y_{p-1}\}$ and the following conditions hold:

(S1) *$f^j_\gamma \in E_G(Y, u^j_\gamma)$ for every $\gamma \in \Gamma$ and $j \in \{1, 2\}$.*

(S2) *$\varphi(f^1_\gamma) = \varphi(f^2_\gamma) = \gamma \in \bar{\varphi}(Y)$ for every $\gamma \in \Gamma$.*

(S3) *$\varphi(e_j) \notin \Gamma$ for $2 \leq j \leq p - 1$.*

(S4) *For every $v \in F(T, e, \varphi)$ there is a colour $\gamma \in \Gamma$ satisfying $\gamma \in \bar{\varphi}(v)$.*

Then G is an elementary graph.

Proof: By Proposition 3.4(b) $V(T)$ is elementary and closed both with respect to φ.

If $\Gamma^d(T, e, \varphi) = \emptyset$ then we conclude from Proposition 3.4(c) that $V(T)$ is also strongly closed with respect to φ. Then Theorem 2.8 implies that G is an elementary graph and we are done. For the rest of the proof we assume $\Gamma^d(T, e, \varphi) \neq \emptyset$. Evidently, this implies $F(T, e, \varphi) \neq \emptyset$ and $s \geq 1$.

Without loss of generality, we may also assume that

(1) $|F(T,e,\varphi)| = s$ and for every vertex $v \in F(T,e,\varphi)$ there is a unique colour $\gamma \in \Gamma$ satisfying $\gamma \in \bar{\varphi}(v)$.

Proof of (1): Since $V(T)$ is elementary with respect to φ, for every $\gamma \in \Gamma$ there is a unique vertex $v \in V(T)$ satisfying $\gamma \in \bar{\varphi}(v)$. Hence, by (S4), there is a subset $\Gamma' \subseteq \Gamma$ with $s' = |\Gamma'| = |F(T,e,\varphi)|$ so that for every vertex $v \in F(T,e,\varphi)$ there is a unique colour $\gamma \in \Gamma'$ satisfying $\gamma \in \bar{\varphi}(v)$.

Now let $\pi : \Gamma \to \Gamma$ a permutation satisfying $\pi(\{\gamma_{s-s'+1},\dots,\gamma_s\}) = \Gamma'$. Then, clearly,

$$T' = (Ty_{p-1}, f^1_{\pi(\gamma_1)}, u^1_{\pi(\gamma_1)}, f^2_{\pi(\gamma_1)}, u^2_{\pi(\gamma_1)}, \dots, f^1_{\pi(\gamma_s)}, u^1_{\pi(\gamma_s)}, f^2_{\pi(\gamma_s)}, u^2_{\pi(\gamma_s)})$$

is a Tashkinov tree with respect to e and φ and, moreover, $(T', e\varphi)$ fulfils conditions similar to (S1)-(S4) just by replacing Γ and Y by Γ' and $Y' = Y \cup \bigcup_{\gamma \in \Gamma \setminus \Gamma'} \{u^1_\gamma, u^2_\gamma\}$, respectively.

Consequently, if (1) doesn't hold for (T,e,φ), Y and Γ, then we only need to consider (T',e,φ), Y' and Γ' instead. This justifies the assumption. $\square$

Since $V(T)$ is elementary with respect to φ, (1) implies that there is a one-to-one correspondence between the s colours from Γ and the s vertices from $F(T,e,\varphi)$, i.e. every vertex $v \in F(T,e,\varphi)$ corresponds to a colour $\gamma \in \Gamma$. This fact shall be used quite often without mentioning it explicitly.

By (S2), all colours which are used on T with respect to φ, belong to $\bar{\varphi}(Y)$. Since $V(T)$ is elementary with respect to φ and $\bar{\varphi}(v) \neq \emptyset$ for every $v \in V(G)$, this implies that $\bar{\varphi}(v) \cap \Gamma^f(T,e,\varphi) \neq \emptyset$ for every vertex $v \in V(T) \setminus Y$. From Lemma 3.8 we then conclude

(2) $F(T,e,\varphi) \subseteq Y$.

For $\gamma \in \Gamma$ let $T - \gamma$ be the Tashkinov tree T without the edges f^1_γ, f^2_γ and the vertices u^1_γ, u^2_γ. Evidently $T - \gamma$ is a Tashkinov tree with respect to e and φ. Furthermore, let

$$U_\gamma = \{u^1_\gamma, u^2_\gamma\}$$

and

$$Z_\gamma = V(T - \gamma) = V(T) \setminus U_\gamma.$$

Since $V(T)$ is elementary and closed with respect to φ, for every $\gamma \in \Gamma$ and every $\alpha \in \bar{\varphi}(Z_\gamma) \setminus \{\gamma\}$ there is an edge $f \in E_G(u^1_\gamma, V(T))$ with $\varphi(f) = \alpha$. Now suppose $f \in E_G(u^1_\gamma, Z_\gamma)$. Then there is a second edge $f' \in E_G(u^2_\gamma, Z_\gamma)$ with $\varphi(f') = \alpha$. Hence $T' = (T - \gamma, f, u^1_\gamma, f', u^2_\gamma)$ is a Tashkinov tree with respect to e and φ satisfying $(T',e,\varphi) \in \mathcal{T}(G)$, $V(T') = V(T)$ and $\gamma \in \Gamma^f(T',e,\varphi)$. Moreover, $\Gamma^d(T',e,\varphi) = \Gamma^d(T,e,\varphi)$, which implies $F(T',e,\varphi) = F(T,e,\varphi)$ and therefore $\gamma \in \bar{\varphi}(F(T',e,\varphi)) \cap \Gamma^f(T',e,\varphi)$, a contradiction to Lemma 3.8. Hence $f \notin E_G(u^1_\gamma, Z_\gamma)$, but $f \in E_G(u^1_\gamma, u^2_\gamma)$ and, therefore, the following statement holds:

(3) $E_G(u^1_\gamma, u^2_\gamma) \cap E_\alpha(e,\varphi) \neq \emptyset$ for every colour $\gamma \in \Gamma$ and every colour $\alpha \in \bar{\varphi}(Z_\gamma)\setminus \{\gamma\}$.

From Proposition 3.4(f) it follows that $|\bar{\varphi}(y_0)| \geq 2$. Since $|\Gamma \cap \bar{\varphi}(y_0)| \leq 1$, this implies that there is a colour

$$\alpha_0 \in \bar{\varphi}(y_0)\setminus\Gamma.$$

For every $\delta \in \Gamma^d(T,e,\varphi)$, let

$$P_\delta = P_{y_0}(\alpha_0, \delta, \varphi).$$

Since $V(T)$ is elementary and closed with respect to φ, and since $\alpha_0 \in \bar{\varphi}(V(T))$ and $\delta \notin \bar{\varphi}(V(T))$, P_δ is a path where one endvertex is y_0 and the other endvertex is some vertex $z_\delta \in V(G)\setminus V(T)$. Let v^0_δ be the last vertex in the linear order $\preceq_{(y_0,P_\delta)}$ that belongs to $V(T)$, and let f^0_δ be the unique edge in $E(P)\cap E_G(v^0_\delta, V(G)\setminus V(T))$. Clearly, this edge belongs to $E_\delta(T,e,\varphi)$. Moreover, since $v^0_\delta \in F(T,e,\varphi)$, by (1) there is a unique colour $\gamma(\delta) \in \Gamma$ satisfying $\gamma(\delta) \in \bar{\varphi}(v^0_\delta)$. Then we claim that

(4) $E_G(U_{\gamma(\delta)}, Z_{\gamma(\delta)}) \cap E_\delta(e,\varphi) = \emptyset$ for every $\delta \in \Gamma^d(T,e,\varphi)$.

Proof of (4): Suppose, on the contrary, that there is a colour $\delta \in \Gamma^d(T,e,\varphi)$ and an edge $g_1 \in E_G(U_{\gamma(\delta)}, Z_{\gamma(\delta)})$ with $\varphi(g_1) = \delta$, say g_1 is incident to $u^1_{\gamma(\delta)}$. From (3) we know that there is an edge $g_2 \in E_G(u^1_{\gamma(\delta)}, u^2_{\gamma(\delta)})$ with $\varphi(g_2) = \alpha_0$.

Clearly, we have $|E_G(U_{\gamma(\delta)}, V(G)\setminus V(T)) \cap E_\delta(e,\varphi)| \leq 1$. Then, evidently, Proposition 3.4(e) implies $|E_G(Z_{\gamma(\delta)}, V(G)\setminus V(T)) \cap E_\delta(e,\varphi)| \geq 2$. Hence, there is an edge $g_3 \in E_G(Z_{\gamma(\delta)}, V(G)\setminus V(T))\setminus\{f^0_\delta\}$ with $\varphi(g_3) = \delta$. Let u_3 be the endvertex of g_3 that belongs to $V(G)\setminus V(T)$.

Now let $P_1 = v^0_\delta P_\delta z_\delta$. Then, clearly, $V(P_1)\cap V(T) = \{v^0_\delta\}$. Furthermore, since $\alpha_0, \gamma(\delta) \in \bar{\varphi}(V(T))$ and $V(T)$ is closed with respect to φ, we can obtain a new coloring $\varphi_1 \in \mathcal{C}_k(G-e)$ from φ by changing the colours α_0 and $\gamma(\delta)$ on all edges in $E_G(V(G)\setminus V(T), V(G)\setminus V(T))$. Then we conclude that $P_1 = P_{v^0_\delta}(\gamma(\delta), \delta, \varphi_1)$. For the coloring $\varphi_2 = \varphi_1/P_1$ we obtain that $\varphi_2 \in \mathcal{C}_k(G-e)$, and $T_1 = T - \gamma(\delta)$ is a Tashkinov tree with respect to e and φ_2 satisfying $V(T_1) = Z_{\gamma(\delta)}$ and $\delta \in \bar{\varphi}_2(v^0_\delta) \subseteq \bar{\varphi}_2(V(T_1))$. Since g_1, g_2, g_3 neither belong to $E_G(V(G)\setminus V(T), V(G)\setminus V(T))$ nor to $E(P_1)$, their colours didn't change, and therefore we have $\varphi_2(g_1) = \varphi_2(g_3) = \delta$ and $\varphi_2(g_2) = \alpha_0$. Then, evidently, $T_2 = (T_1, g_1, u^1_{\gamma(\delta)}, g_2, u^2_{\gamma(\delta)}, g_3, u_3)$ is a Tashkinov tree with respect to e and φ_2 satisfying $|V(T_2)| > |V(T)| = t(G)$, a contradiction. This proves (4). □

Now, for every $\delta \in \Gamma^d(T,e,\varphi)$, let P'_δ defined by

$$P'_\delta = P_{v^0_\delta}(\gamma(\delta), \delta, \varphi).$$

Evidently, P'_δ is a path where one endvertex is v^0_δ and the other endvertex is some vertex $z'_\delta \in V(G)\setminus V(T)$. Then we claim

(5) $V(P'_\delta) \cap V(T) = \{v^0_\delta\}$ for every $\delta \in \Gamma^d(T, e, \varphi)$.

Proof of (5): Suppose, on the contrary, that there is a $\delta \in \Gamma^d(T, e, \varphi)$ with $V(P'_\delta) \cap V(T) \neq \{v^0_\delta\}$. Since $v^0_\delta \in V(P'_\delta) \cap V(T)$, the last vertex v_1 in the linear order $\preceq_{(v^0_\delta, P'_\delta)}$ belonging to $V(T)$ satisfies $v_1 \neq v^0_\delta$. Obviously, $v_1 \in F(T, e, \varphi)$, and from (2) it then follows that $v_1 \in Y$.

Clearly, there is an edge $f_1 \in E_G(v_1, V(G) \setminus V(T))$ with $\varphi(f_1) = \delta$. Let $u_0 \in V(G) \setminus V(T)$ be the second endvertex of f^0_δ and let $u_1 \in V(G) \setminus V(T)$ be the second endvertex of f_1. Furthermore let $P_1 = v^0_\delta P_\delta z_\delta$ and let $P'_1 = v_1 P'_\delta z'_\delta$. Then $V(P_1) \cap V(T) = \{v^0_\delta\}$ and $V(P'_1) \cap V(T) = \{v_1\}$. Since $v_1 \in F(T, e, \varphi)$, by (1) there is a unique $j \in \{1, \ldots, s\}$ with $\gamma_j \in \bar{\varphi}(v_1)$. Moreover, $v_1 \neq v^0_\delta$ implies $\gamma(\delta) \neq \gamma_j$. To simplify notation, let $\gamma = \gamma(\delta)$.

Since $V(T)$ is closed with respect to φ, no edge in $E_G(V(T), V(G) \setminus V(T))$ is coloured with α_0, γ or γ_j with respect to φ. Hence we can obtain two new colourings from φ, the first one $\varphi_1 \in \mathcal{C}_k(G-e)$ by changing the colours α_0 and γ on all edges in $E_G(V(G) \setminus V(T), V(G) \setminus V(T))$, the second one $\varphi'_1 \in \mathcal{C}_k(G-e)$ by changing the colours γ and γ_j on all edges in $E_G(V(G) \setminus V(T), V(G) \setminus V(T))$. Clearly $(T, e, \varphi_1) \in \mathcal{T}(G)$ and $(T, e, \varphi'_1) \in \mathcal{T}(G)$, $\Gamma^f(T, e, \varphi_1) = \Gamma^f(T, e, \varphi'_1) = \Gamma^f(T, e, \varphi)$, $\Gamma^d(T, e, \varphi_1) = \Gamma^d(T, e, \varphi'_1) = \Gamma^d(T, e, \varphi)$ and, moreover, $P_1 = P_{v^0_\delta}(\gamma, \delta, \varphi_1)$ and $P'_1 = P_{v_1}(\gamma_j, \delta, \varphi'_1)$.

For the coloring $\varphi_2 = \varphi_1 / P_1$ we then obtain that $\varphi_2 \in \mathcal{C}_k(G-e)$, and $T_1 = T - \gamma$ is a Tashkinov tree with respect to e and φ_2 satisfying $V(T_1) = Z_\gamma$ and $\delta \in \bar{\varphi}_2(v^0_\delta) \subseteq \bar{\varphi}(V(T_1))$. Since f_1 belongs neither to $E_G(V(G) \setminus V(T), V(G) \setminus V(T))$ nor to $E(P_1)$, its colour didn't change, so we have $\varphi_2(f_1) = \delta$. Moreover, $v_1 \in Y \subseteq V(T_1)$, and therefore $T_2 = (T_1, f_1, u_1)$ is a Tashkinov tree with respect to e and φ_2.

Analogously, for the coloring $\varphi'_2 = \varphi'_1 / P'_1$ we obtain that $\varphi'_2 \in \mathcal{C}_k(G-e)$, and $T'_1 = T - \gamma_j$ is a Tashkinov tree with respect to e and φ'_2 satisfying $V(T'_1) = Z_{\gamma_j}$ and $\delta \in \bar{\varphi}'_2(v_1) \subseteq \bar{\varphi}(v(T'_1))$. Since f^0_δ belongs neither to $E_G(V(G) \setminus V(T), V(G) \setminus V(T))$ nor to P'_1, its colour didn't change, so we have $\varphi'_2(f^0_\delta) = \delta$. Moreover, $v^0_\delta \in Y \subseteq V(T'_1)$, and therefore $T'_2 = (T'_1, f^0_\delta, u_0)$ is a Tashkinov tree with respect to e and φ'_2.

Let $Z = V(T_1) \cap V(T'_1) = V(T) \setminus U_\gamma \setminus U_{\gamma_j}$. Since $\delta \notin \bar{\varphi}(Z)$ and $|Z|$ is odd, also $|E_G(Z, V(G) \setminus Z) \cap E_\delta(e, \varphi)|$ is odd. So beside from f^0_δ and f_1 there is another edge $f_2 \in E_G(Z, V(G) \setminus Z)$ with $\varphi(f_2) = \delta$. Since f_2 has an endvertex in $Z \subseteq V(T)$, but is distinct from f^0_δ or f_1, it neither belongs to $E(P_1)$, to $E(P'_1)$ or to $E_G(V(G) \setminus V(T), V(G) \setminus V(T))$. So none of the recolourings have an effect on f_2, which leads to $\varphi_2(f_2) = \varphi'_2(f_2) = \delta$.

Let u_2 be the endvertex of f_2 that belongs to $V(G) \setminus Z$. We claim that $u_2 \notin U_{\gamma_j}$. Suppose, on the contrary, that $u_2 \in U_{\gamma_j}$, say $u_2 = u^1_{\gamma_j}$. From (3) we then conclude that there is an edge $f' \in E_G(u^1_{\gamma_j}, u^2_{\gamma_j})$ with $\varphi(f') = \alpha_0$. Obviously, we have $\varphi'_2(f') = \alpha_0 \in \bar{\varphi}'_2(V(T'_2))$ and, therefore, $T' = (T'_2, f_2, u^1_{\gamma_j}, f', u^2_{\gamma_j})$ is a Tashkinov tree with respect to e and φ'_2 satisfying $|V(T')| > |V(T)| = t(G)$, a contradiction. This proves the claim, thus we have $u_2 \notin U_{\gamma_j}$. Moreover, from (4) we conclude that $u_2 \notin U_\gamma$ and, therefore, $u_2 \in V(G)\backslash V(T)$. Hence $T_3 = (T_2, f_2, u_2)$ is a Tashkinov tree with respect to e and φ_2 and $T'_3 = (T'_2, f_2, u_2)$ is a Tashkinov tree with respect to e and φ'_2. Since $|V(T_3)| = |V(T'_3)| = |V(T)| = t(G)$, this implies $(T_3, e, \varphi_2), (T'_3, e, \varphi'_2) \in \mathcal{T}(G)$.

From Proposition 3.4(f) it follows that $|\bar{\varphi}(\{y_0, y_1\})| \geq 4$. So there is a colour $\beta \in \bar{\varphi}(\{y_0, y_1\})$ with $\beta \notin \{\alpha_0, \gamma, \gamma_j\}$. Obviously, we also have $\beta \neq \delta$, and therefore the colour β doesn't matter in any of the mentioned recolourings, which leads to $E_\beta(e, \varphi) = E_\beta(e, \varphi_2) = E_\beta(e, \varphi'_2)$. Then, evidently, $\beta \in \bar{\varphi}_2(V(T_3))$. By Proposition 3.4(b), $V(T_3)$ is elementary and closed both with respect to φ_2. Hence there is an edge $f_3 \in E_G(u_2, V(T_3))$ with $\varphi_2(f_3) = \beta$. Clearly, we also have $\varphi(f_3) = \beta$, but since $V(T)$ is closed with respect to φ, the edge f_3 cannot have an endvertex in $V(T)$. Therefore we conclude $f_3 \in E_G(u_2, u_1)$. Moreover, we have $\varphi'_2(f_3) = \beta \in \bar{\varphi}'_3(V(T'_3))$ and hence $T'_4 = (T_3, f_3, u_1)$ is a Tashkinov tree with respect to e and φ'_2 satisfying $|V(T'_4)| > |V(T)| = t(G)$, a contradiction. This proves (5). □

Now, for every $\delta \in \Gamma^d(T, e, \varphi)$, we define a new coloring $\varphi_\delta \in \mathcal{C}_k(G - e)$ by

$$\varphi_\delta = \varphi / P'_\delta.$$

From (5) we then obtain the following.

(6) For every $\delta \in \Gamma^d(T, e, \varphi)$ the coloring $\varphi_\delta \in \mathcal{C}_k(G - e)$ satisfies:

- $\varphi_\delta(f^0_\delta) = \gamma(\delta)$,
- $\varphi_\delta(f) = \varphi(f)$ for every edge $f \in E_{G-e}(V(T), V(G)) \setminus \{f^0_\delta\}$,
- $\varphi_\delta(f) = \varphi(f)$ for every edge $f \in E(G - e) \setminus E(P'_\delta)$,
- $\bar{\varphi}_\delta(v^0_\delta) = \bar{\varphi}(v^0_\delta) \setminus \{\gamma(\delta)\} \cup \{\delta\}$,
- $\bar{\varphi}_\delta(v) = \bar{\varphi}(v)$ for every vertex $v \in V(T) \setminus \{v^0_\delta\}$,
- $\bar{\varphi}_\delta(v) = \bar{\varphi}(v)$ for every vertex $v \in V(G) \setminus V(P'_\delta)$.

Next we claim that

(7) $|E_G(Z_{\gamma(\delta)}, V(G) \setminus V(T)) \cap E_\delta(e, \varphi)| = 3$ for every $\delta \in \Gamma^d(T, e, \varphi)$.

Proof of (7): Suppose, on the contrary, that there is a $\delta \in \Gamma^d(T, e, \varphi)$ satisfying $|E_G(Z_{\gamma(\delta)}, V(G) \setminus V(T)) \cap E_\delta(e, \varphi)| \neq 3$.

Consider the case $|E_G(Z_{\gamma(\delta)}, V(G) \setminus V(T)) \cap E_\delta(e, \varphi)| > 3$. Then beside f^0_δ there are another three edges $g_1, g_2, g_3 \in E_G(Z_{\gamma(\delta)}, V(G) \setminus V(T))$ with endvertices $z_1, z_2, z_3 \in V(G) \setminus V(T)$ and $\varphi(g_1) = \varphi(g_2) = \varphi(g_3) = \delta$. From (6) it then follows that $\varphi_\delta(g_1) = \varphi_\delta(g_2) = \varphi_\delta(g_3) = \delta$. Hence $T_1 = (T - \gamma(\delta), g_1, z_1, g_2, z_2, g_3, z_3)$ is a Tashkinov tree with respect to e and φ_δ satisfying $|V(T_1)| > |V(T)| = t(G)$, a contradiction. Consequently, we have $|E_G(Z_{\gamma(\delta)}, V(G) \setminus V(T)) \cap E_\delta(e, \varphi)| < 3$.

From (4) we know that $|E_G(Z_{\gamma(\delta)}, U_{\gamma(\delta)}) \cap E_\delta(e,\varphi)| = 0$. Therefore we have $|E_G(Z_{\gamma(\delta)}, V(G) \setminus Z_{\gamma(\delta)}) \cap E_\delta(e,\varphi)| = |E_G(Z_{\gamma(\delta)}, V(G) \setminus V(T)) \cap E_\delta(e,\varphi)| < 3$. Since $\delta \notin \bar{\varphi}(Z_{\gamma(\delta)})$ and $|Z_{\gamma(\delta)}|$ is odd, this implies that $|E_G(Z_{\gamma(\delta)}, V(G) \setminus Z_{\gamma(\delta)}) \cap E_\delta(e,\varphi)|$ is odd, too, which leads to $|E_G(Z_{\gamma(\delta)}, V(G) \setminus Z_{\gamma(\delta)}) \cap E_\delta(e,\varphi)| = 1$. Consequently, we have $E_G(Z_{\gamma(\delta)}, V(G) \setminus Z_{\gamma(\delta)}) \cap E_\delta(e,\varphi) = \{f_\delta^0\}$.

By (3) we have $E_G(Z_{\gamma(\delta)}, U_{\gamma(\delta)}) \cap E_{\alpha_0}(e,\varphi) = \emptyset$. Since $\alpha_0 \in \bar{\varphi}(V(T))$ and $V(T)$ is closed with respect to φ, this implies $E_G(Z_{\gamma(\delta)}, V(G) \setminus Z_{\gamma(\delta)}) \cap E_{\alpha_0}(e,\varphi) = \emptyset$. Hence $E_G(Z_{\gamma(\delta)}, V(G) \setminus Z_{\gamma(\delta)}) \cap E(P_\delta) = \{f_\delta^0\}$. For the subpath $P_1 = y_0 P_\delta v_\delta^0$ this means $V(P_1) \subseteq Z_{\gamma(\delta)}$. Since v_δ^0 is the last vertex in the linear order $\preceq_{(y_0,P_\delta)}$ that belongs to $V(T)$, we conclude $V(P_\delta) \cap V(T) \subseteq Z_{\gamma(\delta)}$.

Then especially the vertices $u^1_{\gamma(\delta)}, u^2_{\gamma(\delta)}$ don't belong to P_δ. Hence, the chain $P_2 = P_{u^1_{\gamma(\delta)}}(\alpha_0, \delta, \varphi)$ is vertex disjoint to P_δ and, moreover, $V(P_2) \cap V(T) \subseteq U_{\gamma(\delta)}$. Then, evidently, $E(P_2) \cap E(T) = \emptyset$ and hence T is a Tashkinov tree with respect to e and the coloring $\varphi_2 = \varphi/P_2$. Since $|E_G(Z_{\gamma(\delta)}, V(G) \setminus V(T)) \cap E_\delta(e,\varphi)| = 1$, Proposition 3.4(e) implies that there are two edges $g_4 \in E_G(u_\delta^1, V(G) \setminus V(T))$ and $g_5 \in E_G(u_\delta^2, V(G) \setminus V(T))$ with $\varphi(g_4) = \varphi(g_5) = \delta$. Evidently $g_4 \in E(P_2)$ and $\varphi_2(g_4) = \alpha_0 \in \bar{\varphi}_2(y_0)$. If u_4 is the endvertex of g_4 belonging to $V(G) \setminus V(T)$, then $T_2 = (T, g_4, u_4)$ is a Tashkinov tree with respect to e and φ_2 satisfying $|V(T_2)| > |V(T)| = t(G)$, a contradiction. This proves the claim. $\square$

For every $\delta \in \Gamma^d(T,e,\varphi)$ we know from (7) that beside f_δ^0 there are two other edges $f_\delta^1, f_\delta^2 \in E_G(Z_{\gamma(\delta)}, V(G) \setminus V(T))$ with $\varphi(f_\delta^1) = \varphi(f_\delta^2) = \delta$. For $j = 1, 2$ let $f_\delta^j \in E_G(v_\delta^j, u_\delta^j)$ where $v_\delta^1, v_\delta^2 \in Z_{\gamma(\delta)}$ and $u_\delta^1, u_\delta^2 \in V(G) \setminus V(T)$. Furthermore, let

$$U_\delta = \{u_\delta^1, u_\delta^2\}.$$

By (5) we have $f_\delta^1, f_\delta^2 \notin E(P'_\delta)$ and therefore $u_\delta^1, u_\delta^2 \notin V(P'_\delta)$. Hence (6) implies:

(8) $\varphi_\delta(f) = \varphi(f)$ for every $\delta \in \Gamma^d(T,e,\varphi)$ and every $f \in E_G(U_\delta, V(G))$.

In particular, for every $\delta \in \Gamma^d(T,e,\varphi)$ this leads to $\varphi_\delta(f_\delta^1) = \varphi_\delta(f_\delta^2) = \delta$. From $\delta \in \bar{\varphi}_\delta(v_\delta^0)$ then follows that T_δ, defined by

$$T_\delta = (T - \gamma(\delta), f_\delta^1, u_\delta^1, f_\delta^2, u_\delta^2),$$

is a Tashkinov tree with respect to e and φ_δ satisfying $|V(T_\delta)| = |V(T)|$. Therefore, we obtain that

(9) $(T_\delta, e, \varphi_\delta) \in \mathcal{T}(G)$ for every $\delta \in \Gamma^d(T,e,\varphi)$.

Since $V(T)$ is closed with respect to φ, for every $\delta \in \Gamma^d(T,e,\varphi)$ and every $\alpha \in \bar{\varphi}(Z_{\gamma(\delta)}) \setminus \{\gamma(\delta)\}$ we have $E_G(Z_{\gamma(\delta)}, V(G) \setminus V(T)) \cap E_\alpha(e,\varphi) = \emptyset$. This implies $E_G(Z_{\gamma(\delta)}, U_\delta) \cap E_\alpha(e,\varphi) = \emptyset$ and, moreover, by (8), $E_G(Z_{\gamma(\delta)}, U_\delta) \cap E_\alpha(e,\varphi_\delta) =$

$\emptyset$. Since $\alpha \in \bar{\varphi}_\delta(Z_{\gamma(\delta)}) \subseteq \bar{\varphi}_\delta(V(T_\delta))$ and since, by Proposition 3.4(b), $V(T_\delta)$ is elementary and closed with respect to φ_δ, there must be an edge between u_δ^1 and u_δ^2 coloured with α with respect to φ_δ and by (8) also with respect to φ. Therefore, we have

(10) $E_G(u_\delta^1, u_\delta^2) \cap E_\alpha(e, \varphi) \neq \emptyset$ for every $\delta \in \Gamma^d(T, e, \varphi)$ and every $\alpha \in \bar{\varphi}(Z_{\gamma(\delta)}) \setminus \{\gamma(\delta)\}$.

Further we claim that the following two statements are true.

(11) $E_G(u_{\gamma(\delta)}^1, u_{\gamma(\delta)}^2) \cap E_\delta(e, \varphi) \neq \emptyset$ for every $\delta \in \Gamma^d(T, e, \varphi)$.

(12) $E_G(u_\delta^1, u_\delta^2) \cap E_{\gamma(\delta)}(e, \varphi) \neq \emptyset$ for every $\delta \in \Gamma^d(T, e, \varphi)$.

Proof of (11): Let $\delta \in \Gamma^d(T, e, \varphi)$. We have $|Z_{\gamma(\delta)}| = p + 2s - 2$ and therefore, by Proposition 3.4(f), $|\bar{\varphi}(Z_{\gamma(\delta)})| \geq p + 2s$. Since there are at most $p - 2 + s$ colours used on T with respect to φ, there is a colour $\beta \in \bar{\varphi}(Z_{\gamma(\delta)}) \cap \Gamma^f(T, e, \varphi)$.

Let $v \in Z_{\gamma(\delta)}$ be the unique vertex with $\beta \in \bar{\varphi}(v)$, and let $P = P_v(\beta, \delta, \varphi)$. From (7) and Proposition 3.6 we conclude that P is a path having one endvertex v and another endvertex $z \in V(G) \setminus V(T)$ satisfying $E(P) \cap E_G(Z_{\gamma(\delta)}, V(G) \setminus V(T)) = \{f_\delta^0, f_\delta^1, f_\delta^2\}$. By (2) we have $F(T, e, \varphi) \subseteq Y \subseteq Z_{\gamma(\delta)}$, so for the last vertex v' in the linear order $\preceq_{(v,P)}$ belonging to $V(T)$ we conclude $v' \in \{v_\delta^0, v_\delta^1, v_\delta^2\}$.

From (10) follows that there is an edge $g \in E_G(u_\delta^1, u_\delta^2)$ with $\varphi(g) = \beta$. Hence the path $P_1 = P(v_\delta^1, f_\delta^1, u_\delta^1, g, u_\delta^2, f_\delta^2, v_\delta^2)$ is a subpath of P, and therefore $v' \notin \{v_\delta^1, v_\delta^2\}$, but $v' = v_\delta^0$.

By (3), we have $E_G(U_{\gamma(\delta)}, Z_{\gamma(\delta)}) \cap E_\beta(e, \varphi) = \emptyset$, and from (4) follows that $E_G(U_{\gamma(\delta)}, Z_{\gamma(\delta)}) \cap E_\delta(e, \varphi) = \emptyset$. Hence, for the subpath $P_2 = vPv_\delta^0$ we conclude $V(P_2) \subseteq Z_{\gamma(\delta)} \cup U_\delta$. Further, for $P_3 = v_\delta^0 P z$ we clearly have $V(P_3) \cap V(T) = \{v_\delta^0\}$ and hence $V(P) \cap U_{\gamma(\delta)} = \emptyset$. Then from Proposition 3.6(b) follows that $E_G(U_{\gamma(\delta)}, V(G) \setminus V(T)) \cap E_\delta(e, \varphi) = \emptyset$. Since $E_G(U_{\gamma(\delta)}, Z_{\gamma(\delta)}) \cap E_\delta(e, \varphi) = \emptyset$ and $\delta \notin \bar{\varphi}(V(T))$, we conclude $E_G(u_{\gamma(\delta)}^1, u_{\gamma(\delta)}^2) \cap E_\delta(e, \varphi) \neq \emptyset$. This proves the claim. □

Proof of (12): Let $\delta \in \Gamma^d(T, e, \varphi)$. We have $|Z_{\gamma(\delta)}| = p + 2s - 2$ and therefore, by Proposition 3.4(f), $|\bar{\varphi}_\delta(Z_{\gamma(\delta)})| \geq p + 2s$. Since there are at most $p - 2 + s$ colours used on T_δ with respect to φ_δ, there is a colour $\beta \in \bar{\varphi}_\delta(Z_{\gamma(\delta)}) \cap \Gamma^f(T, e, \varphi_\delta)$. Let $v \in Z_{\gamma(\delta)}$ be the unique vertex with $\beta \in \bar{\varphi}_\delta(v)$.

By (6) we have $\varphi_\delta(f_\delta^0) = \varphi_\delta(f_{\gamma(\delta)}^1) = \varphi_\delta(f_{\gamma(\delta)}^2) = \gamma(\delta)$ and therefore $\gamma(\delta) \in \Gamma^d(T_\delta, e, \varphi_\delta)$. Moreover, beside this three edges there can be no further edge in $E_G(Z_{\gamma(\delta)}, V(G) \setminus V(T_\delta))$ coloured with $\gamma(\delta)$ with respect to φ_δ. Otherwise such an edge f would, by (6), satisfy $\varphi(f) = \gamma(\delta)$ and it would belong to $E_G(V(T), V(G) \setminus V(T))$. This would contradict the fact that $V(T)$ is closed with respect to φ. Hence, we have $E_G(Z_{\gamma(\delta)}, V(G) \setminus V(T_\delta)) \cap E_{\gamma(\delta)}(e, \varphi_\delta) = \{f_\delta^0, f_{\gamma(\delta)}^1, f_{\gamma(\delta)}^2\}$. From this and

Proposition 3.6 we conclude that $P = P_v(\beta, \gamma(\delta), \varphi_\delta)$ is a path having one endvertex v and another endvertex $z \in V(G)\backslash V(T_\delta)$ satisfying $E(P) \cap E_G(Z_{\gamma(\delta)}, V(G)\backslash V(T_\delta)) = \{f_\delta^0, f_{\gamma(\delta)}^1, f_{\gamma(\delta)}^2\}$.

By Proposition 3.4(f) we have $\bar{\varphi}_\delta(u_\delta^1) \neq \emptyset$ and $\bar{\varphi}_\delta(u_\delta^2) \neq \emptyset$. Since no colour in $\bar{\varphi}_\delta(U_\delta)$ is used on T_δ with respect to φ_δ, Lemma 3.8 implies $F(T_\delta, e, \varphi_\delta) \subseteq Z_{\gamma(\delta)}$. Hence, for the last vertex v' in the linear order $\preceq_{(v,P)}$ belonging to $V(T)$ we conclude $v' \in \{v_\delta^0, v_1, v_2\}$, where v_1, v_2 are the two endvertices of $f_{\gamma(\delta)}^1, f_{\gamma(\delta)}^2$ belonging to $Z_{\gamma(\delta)}$.

From (3) follows that there is an edge $g \in E_G(u_{\gamma(\delta)}^1, u_{\gamma(\delta)}^2)$ with $\varphi(g) = \beta$ and, by (6), also $\varphi_\delta(g) = \beta$. Hence, $P_1 = P(v_1, f_{\gamma(\delta)}^1, u_{\gamma(\delta)}^1, g, u_{\gamma(\delta)}^2, f_{\gamma(\delta)}^2, v_2)$ is a subpath of P, and therefore $v' \notin \{v_1, v_2\}$, but $v' = v_\delta^0$.

Since $V(T)$ is closed with respect to φ, we clearly have $E_G(U_\delta, Z_{\gamma(\delta)}) \cap E_\beta(e, \varphi) = \emptyset$ and $E_G(U_\delta, Z_{\gamma(\delta)}) \cap E_{\gamma(\delta)}(e, \varphi) = \emptyset$. Therefore, by (6), we have $E_G(U_\delta, Z_{\gamma(\delta)}) \cap E_\beta(e, \varphi_\delta) = \emptyset$ and $E_G(U_\delta, Z_{\gamma(\delta)}) \cap E_{\gamma(\delta)}(e, \varphi_\delta) = \emptyset$. Hence for the subpath $P_2 = vPv_\delta^0$ we conclude $V(P_2) \subseteq Z_{\gamma(\delta)} \cup U_{\gamma(\delta)}$. Furthermore, for $P_3 = v_\delta^0 P z$ we clearly have $V(P_3) \cap V(T_\delta) = \{v_\delta^0\}$ and, therefore, $V(P) \cap U_\delta = \emptyset$. Then from Proposition 3.6(b) follows that $E_G(U_\delta, V(G) \setminus V(T_\delta)) \cap E_{\gamma(\delta)}(e, \varphi_\delta) = \emptyset$. Since we have $E_G(U_\delta, Z_{\gamma(\delta)}) \cap E_{\gamma(\delta)}(e, \varphi_\delta) = \emptyset$ and $\gamma(\delta) \notin \bar{\varphi}_\delta(V(T_\delta))$, we conclude $E_G(u_\delta^1, u_\delta^2) \cap E_{\gamma(\delta)}(e, \varphi_\delta) \neq \emptyset$. From (8) then follows $E_G(u_\delta^1, u_\delta^2) \cap E_{\gamma(\delta)}(e, \varphi) \neq \emptyset$. This proves the claim. □

Further we claim, that the following two statements are true:

(13) $E_G(u_{\gamma(\delta)}^1, u_{\gamma(\delta)}^2) \cap E_\alpha(e, \varphi) \neq \emptyset$ for every $\delta \in \Gamma^d(T, e, \varphi)$ and every $\alpha \in \bar{\varphi}(U_\delta)$.

(14) $E_G(u_\delta^1, u_\delta^2) \cap E_\alpha(e, \varphi) \neq \emptyset$ for every $\delta \in \Gamma^d(T, e, \varphi)$ and every $\alpha \in \bar{\varphi}(U_{\gamma(\delta)})$.

Proof of (13): Let $\delta \in \Gamma^d(T, e, \varphi)$ and let $\alpha \in \bar{\varphi}(U_\delta)$. Then clearly $\alpha \neq \delta$ and by (12) also $\alpha \neq \gamma(\delta)$. Hence $E_\alpha(e, \varphi) = E_\alpha(e, \varphi_\delta)$, and therefore $\alpha \in \bar{\varphi}_\delta(U_\delta)$. Moreover $\alpha_0 \in \bar{\varphi}_\delta(y_0)$, and $V(T_\delta)$ is elementary with respect to φ_δ. Hence, we have $\alpha \neq \alpha_0$. Since $V(T_\delta)$ is also closed with respect to φ_δ, for $P = P_{u_{\gamma(\delta)}^1}(\alpha_0, \alpha, \varphi) = P_{u_{\gamma(\delta)}^1}(\alpha_0, \alpha, \varphi_\delta)$ we have $V(P) \subseteq V(G) \setminus V(T_\delta)$. This implies $V(P) \cap V(T) \subseteq U_{\gamma(\delta)}$ and therefore $E(P) \cap E(T) = \emptyset$. Then T is a Tashkinov tree with respect to e and the coloring $\varphi' = \varphi/P$, and from $y_0 \notin V(P)$ we conclude $\alpha_0 \in \bar{\varphi}'(V(T))$.

From (3) we know that $E_G(u_{\gamma(\delta)}^1, u_{\gamma(\delta)}^2) \cap E_{\alpha_0}(e, \varphi) \neq \emptyset$ and, therefore, we have $u_{\gamma(\delta)}^2 \in V(P)$. If there is an edge $g \in E_G(U_{\gamma(\delta)}, z)$ for a vertex $z \in V(G) \setminus V(T)$ and $\varphi(g) = \alpha$, then we have $\varphi'(g) = \alpha_0 \in \bar{\varphi}'(V(T))$, and $T' = (T, g, z)$ is a Tashkinov tree with respect to e and φ', satisfying $|V(T')| > |V(T)| = t(G)$, a contradiction. If there is an edge $g \in E_G(U_{\gamma(\delta)}, Z_{\gamma(\delta)})$ with $\varphi(g) = \alpha$, then we have $\varphi_\delta(g) = \alpha \in \bar{\varphi}_\delta(V(T_\delta))$, a contradiction, too, because $V(T_\delta)$ is closed with respect to φ_δ. Consequently, we have $E_G(U_{\gamma(\delta)}, V(G) \setminus U_{\gamma(\delta)}) \cap E_\alpha(e, \varphi) = \emptyset$. Since $V(T)$ is elementary with respect to φ, we conclude that there is an edge $g \in E_G(u_{\gamma(\delta)}^1, u_{\gamma(\delta)}^2)$ with $\varphi(g) = \alpha$. This proves the claim. □

Proof of (14): Let $\delta \in \Gamma^d(T,e,\varphi)$ and let $\alpha \in \bar{\varphi}(U_{\gamma(\delta)})$. Then, clearly, we have $\alpha \neq \gamma(\delta)$ and, by (11), also $\alpha \neq \delta$. Hence $E_\alpha(e,\varphi) = E_\alpha(e,\varphi_\delta)$. Moreover, $\alpha_0 \in \bar{\varphi}_\delta(y_0)$ and $V(T)$ is elementary with respect to φ, and hence $\alpha \neq \alpha_0$. Since $V(T)$ is also closed with respect to φ, for $P = P_{u_\delta^1}(\alpha_0,\alpha,\varphi) = P_{u_\delta^1}(\alpha_0,\alpha,\varphi_\delta)$ we have $V(P) \subseteq V(G) \setminus V(T)$. This implies $V(P) \cap V(T_\delta) \subseteq U_\delta$ and therefore $E(P) \cap E(T_\delta) = \emptyset$. Then T_δ is a Tashkinov tree with respect to e and the coloring $\varphi' = \varphi_\delta/P$, and from $y_0 \notin V(P)$ we conclude $\alpha_0 \in \bar{\varphi}'(V(T_\delta))$.

From (10) we know $E_G(u_\delta^1,u_\delta^2) \cap E_{\alpha_0}(e,\varphi) \neq \emptyset$, and therefore $u_\delta^2 \in V(P)$. If there was an edge $g \in E_G(U_\delta,z)$ with $z \in V(G)\setminus V(T_\delta)$ and $\varphi_\delta(g) = \alpha$, then we would have $\varphi'(g) = \alpha_0 \in \bar{\varphi}'(V(T_\delta))$, and $T' = (T_\delta,g,z)$ would be a Tashkinov tree with respect to e and φ' satisfying $|V(T')| > |V(T_\delta)| = t(G)$, a contradiction. If there was an edge $g \in E_G(U_\delta, Z_{\gamma(\delta)})$ with $\varphi_\delta(g) = \alpha$, then we would have $\varphi(g) = \alpha \in \bar{\varphi}(V(T))$, a contradiction, too, because $V(T)$ is closed with respect to φ. Consequently, we have $E_G(U_\delta, V(G)\setminus U_\delta) \cap E_\alpha(e,\varphi_\delta) = \emptyset$. Since $V(T_\delta)$ is elementary with respect to φ_δ, we conclude that there is an edge $g \in E_G(u_\delta^1,u_\delta^2)$ with $\varphi_\delta(g) = \varphi(g) = \alpha$. This proves the claim. □

Next we claim

(15) $U_\delta \subseteq A(T,e,\varphi)$ for every $\delta \in \Gamma^d(T,e,\varphi)$.

Proof of (15): Let $\delta \in \Gamma^d(T,e,\varphi)$, let $\alpha \in \bar{\varphi}(U_\delta)$ and let $\beta \in \Gamma^f(T,e,\varphi) \setminus \{\alpha\}$. Clearly, we have $\alpha \neq \delta$ and, by (12), also $\alpha \neq \gamma(\delta)$. Since neither δ nor $\gamma(\delta)$ is a free colour with respect to (T,e,φ), we also have $\beta \notin \{\delta, \gamma(\delta)\}$. Hence $E_\alpha(e,\varphi) = E_\alpha(e,\varphi_\delta)$ and $E_\beta(e,\varphi) = E_\beta(e,\varphi_\delta)$.

Let $u \in U_\delta$ be the unique vertex with $\alpha \in \bar{\varphi}(u) = \bar{\varphi}_\delta(u)$, and let $v \in V(T)$ be the unique vertex with $\beta \in \bar{\varphi}(v) = \bar{\varphi}_\delta(v)$. Moreover, let $P = P_u(\alpha,\beta,\varphi) = P_u(\alpha,\beta,\varphi_\delta)$. Obviously, P is a path having u as an endvertex. We now have to show that v is the second endvertex of P, this would imply $u \in A(T,e,\varphi)$.

In the case $v \in Z_{\gamma(\delta)}$ we have $v \in V(T_\delta)$ and therefore $\beta \in \bar{\varphi}_\delta(V(T_\delta))$. Since also $\alpha \in \bar{\varphi}_\delta(V(T_\delta))$, Theorem 3.3(e) implies that v is the second endvertex of P.

In the other case, we have $v \in U_{\gamma(\delta)}$. By (14) we have $E_G(u_\delta^1,u_\delta^2) \cap E_\beta(e,\varphi) \neq \emptyset$ and therefore $U_\delta \subseteq V(P)$. Since $\alpha \in \bar{\varphi}_\delta(u)$ and $V(T_\delta)$ is elementary and closed with respect to φ_δ, there is an edge $f_2 \in E_G(U_\delta, Z_{\gamma(\delta)})$ with $\varphi_\delta(f_2) = \alpha$ and, moreover, $E_G(Z_{\gamma(\delta)}, V(G)\setminus Z_{\gamma(\delta)}) \cap E_\alpha(e,\varphi_\delta) = \{f_2\}$. Therefore $E_G(Z_{\gamma(\delta)}, V(G)\setminus Z_{\gamma(\delta)}) \cap E_\alpha(e,\varphi) = \{f_2\}$ and $f_2 \in E(P)$. Since $\beta \in \bar{\varphi}(U_{\gamma(\delta)})$ and $V(T)$ is elementary and closed with respect to φ, there is an edge $f_3 \in E_G(U_{\gamma(\delta)}, Z_{\gamma(\delta)})$ with $\varphi(f_3) = \beta$ and, moreover, $E_G(Z_{\gamma(\delta)}, V(G)\setminus Z_{\gamma(\delta)}) \cap E_\beta(e,\varphi) = \{f_3\}$. Since we have $f_2 \in E(P)$ and $\alpha, \beta \notin \bar{\varphi}(Z_{\gamma(\delta)})$ and since f_2, f_3 are the only two edges in $E_G(Z_{\gamma(\delta)}, V(G)\setminus Z_{\gamma(\delta)})$ coloured with α or β with respect to φ, we conclude that $f_3 \in E(P)$. By (13) there is an edge $f_4 \in E_G(u_{\gamma(\delta)}^1, u_{\gamma(\delta)}^2)$ with $\varphi(f_4) = \alpha$. This implies $f_4 \in E(P)$ and therefore $U_{\gamma(\delta)} \subseteq V(P)$. Hence we have $v \in V(P)$, so v must be the second endvertex of P.

In both cases P is a path with endvertices u and v. Hence, by definition, we have $u \in A(T, e, \varphi)$ and the claim is proved. □

Now let

$$X = V(T) \cup \bigcup_{\delta \in \Gamma^d(T,e,\varphi)} U_\delta.$$

Then, by (15), we have $X \subseteq V(T) \cup A(T, e, \varphi)$. Hence Proposition 3.5 implies

(16) X is elementary with respect to φ.

The aim is to show that X is also closed with respect to φ. To do this, we first claim

(17) $X_\delta = V(T) \cup U_\delta$ is closed with respect to φ for every $\delta \in \Gamma^d(T, e, \varphi)$.

Proof of (17): Let $\delta \in \Gamma^d(T, e, \varphi)$ and let $\alpha \in \bar{\varphi}(X_\delta)$. We have to show that $E_G(X_\delta, V(G) \setminus X_\delta) \cap E_\alpha(e, \varphi) = \emptyset$.

If $\alpha \in \bar{\varphi}(V(T))$ we conclude from (10), (12) and (14) that $E_G(u^1_\delta, u^2_\delta) \cap E_\alpha(e, \varphi) \neq \emptyset$ and therefore $E_G(U_\delta, V(G) \setminus X_\delta) \cap E_\alpha(e, \varphi) = \emptyset$. Moreover, since $V(T)$ is closed with respect to φ, we also have $E_G(V(T), V(G) \setminus X_\delta) \cap E_\alpha(e, \varphi) = \emptyset$. Hence, we conclude $E_G(X_\delta, V(G) \setminus X_\delta) \cap E_\alpha(e, \varphi) = \emptyset$.

If $\alpha \in \bar{\varphi}(U_\delta)$ then, clearly, we have $\alpha \neq \delta$ and, by (12), we also have $\alpha \neq \gamma(\delta)$. Hence $E_\alpha(e, \varphi) = E_\alpha(e, \varphi_\delta)$. Consequently, $\alpha \in \bar{\varphi}_\delta(V(T_\delta))$, and since $V(T_\delta)$ is closed with respect to φ_δ, we conclude $E_G(V(T_\delta), V(G) \setminus X_\delta) \cap E_\alpha(e, \varphi) = E_G(V(T_\delta), V(G) \setminus X_\delta) \cap E_\alpha(e, \varphi_\delta) = \emptyset$. Moreover, from (13) we know that $E_G(u^1_{\gamma(\delta)}, u^2_{\gamma(\delta)}) \cap E_\alpha(e, \varphi) \neq \emptyset$ and therefore $E_G(U_{\gamma(\delta)}, V(G) \setminus X_\delta) \cap E_\alpha(e, \varphi) = \emptyset$. Hence, we conclude $E_G(X_\delta, V(G) \setminus X_\delta) \cap E_\alpha(e, \varphi) = \emptyset$.

In any case, we have $E_G(X_\delta, V(G) \setminus X_\delta) \cap E_\alpha(e, \varphi) = \emptyset$. This proves the claim. □

Since, by (10), we have $E_G(u^1_\delta, u^2_\delta) \cap E_{\alpha_0}(e, \varphi) \neq \emptyset$ for all $\delta \in \Gamma^d(T, e, \varphi)$, we easily conclude the following.

(18) For any $\delta, \delta' \in \Gamma^d(T, e, \varphi)$ the sets U_δ and $U_{\delta'}$ are either equal or disjoint.

Now we can show that

(19) X is closed with respect to φ.

Proof of (19): Suppose, on the contrary, that X is not closed with respect to φ, i.e. there exists a colour $\alpha \in \bar{\varphi}(X)$ and an edge $f \in E_G(X, V(G) \setminus X)$ with $\varphi(f) = \alpha$. Then, clearly, there is a colour $\delta \in \Gamma^d(T, e, \varphi)$ satisfying $f \in E_G(V(T) \cup U_\delta, V(G) \backslash X)$. Since, by (17), $V(T) \cup U_\delta$ is closed with respect to φ, we conclude that $\alpha \in \bar{\varphi}(X \setminus V(T) \setminus U_\delta)$ and therefore, by (18), $\alpha \in \bar{\varphi}(U_{\delta'})$ for a colour $\delta' \in \Gamma^d(T, e, \varphi)$ with $U_\delta \cap U_{\delta'} = \emptyset$.

Since, by (17), also $V(T) \cup U_{\delta'}$ is closed with respect to φ, we have $f \notin E_G(V(T) \cup U_{\delta'}, V(G) \setminus X)$. In particular, this means $f \notin E_G(V(T), V(G) \setminus X)$ and, therefore, we conclude $f \in E_G(u, v)$ for two vertices $u \in U_\delta$ and $v \in V(G) \setminus X$. Now let $P = P_u(\alpha_0, \alpha, \varphi)$. Since $\alpha_0, \alpha \in \bar{\varphi}(V(T) \cup U_{\delta'})$ and $V(T) \cup U_{\delta'}$ closed with respect to φ, this implies $V(P) \cap V(T) = \emptyset$. Hence we have $E(P) \cap E(T_\delta) = \emptyset$.

By (16) X is elementary with respect to φ. Since $\alpha \in \bar{\varphi}(U_{\delta'})$ and $\Gamma \subseteq \bar{\varphi}(V(T))$, we conclude $\alpha \notin \Gamma$. Moreover, since, by (17), $V(T) \cup U_{\delta'}$ is closed with respect to φ and since $f_\delta^1 \in E_G(V(T), V(G) \setminus V(T) \setminus U_{\delta'})$, we also conclude that $\alpha \neq \delta$. Moreover, we also have $\alpha_0 \notin \Gamma$ and $\alpha_0 \neq \delta$. Evidently, we conclude $E_\alpha(e, \varphi) = E_\alpha(e, \varphi_\delta)$ and $E_{\alpha_0}(e, \varphi) = E_{\alpha_0}(e, \varphi_\delta)$, which especially implies $P = P_u(\alpha_0, \alpha, \varphi_\delta)$. From $E(P) \cap E(T_\delta) = \emptyset$ then follows that T_δ is a Tashkinov tree with respect to e and $\varphi' = \varphi_\delta / P$. Since $f \in E(P)$, we have $\varphi'(f) = \alpha_0 \in \bar{\varphi}'(V(T_\delta)$. Hence $T' = (T_\delta, f, v)$ is a Tashkinov tree with respect to e and φ' satisfying $|V(T')| > |V(T_\delta)| = t(G)$, a contradiction. This proves the claim. □

Next, we claim the following:

(20) If $\alpha \notin \bar{\varphi}(X)$ and $P = P_{y_0}(\alpha_0, \alpha, \varphi)$, then $|E(P) \cap E_G(X, V(G) \setminus X)| = 1$.

Proof of (20): By (16), X is elementary with respect to φ and, since $\alpha_0 \in \bar{\varphi}(y_0)$, we know that P is a path with one endvertex y_0 and another endvertex $z \in V(G) \backslash X$. Evidently, there is a last vertex v in the linear order $\preceq_{(y_0, P)}$ that belongs to X and there is an edge in $g \in E_G(v, V(G) \backslash X)$ with $\varphi(g) = \alpha$. For the subpath $P_1 = y_0 P v$ of P we have to show that $V(P_1) \subseteq X$, this would complete the proof of (20). To do this, we distinct the following cases.

Case 1: $v \in V(T)$ and $\alpha \notin \Gamma^d(T, e, \varphi)$. Then we have $E_G(V(T), V(G) \backslash V(T)) \cap E_\alpha(e, \varphi) = \{g\}$. Since $\alpha_0 \in \bar{\varphi}(V(T))$ and $V(T)$ is closed with respect to φ, we conclude that $E(P) \cap E_G(V(T), V(G) \backslash V(T) = \{g\}$ and, therefore, $V(P_1) \subseteq V(T) \subseteq X$.

Case 2: $v \in V(T)$ and $\alpha \in \Gamma^d(T, e, \varphi)$. Then from (7) and (11) we conclude that $E_G(V(T), V(G) \setminus V(T)) \cap E_\alpha(e, \varphi) = \{f_\alpha^0, f_\alpha^1, f_\alpha^2\}$. Hence, we have $E_G(X_\alpha, V(G) \setminus X_\alpha) = \{f_\alpha^0\}$, which implies $g = f_\alpha^0$. Since $\alpha_0 \in \bar{\varphi}(X_\alpha)$ and, by (17), X_δ is closed with respect to φ, it follows that $V(P_1) \subseteq X_\alpha \subseteq X$.

Case 3: $v \notin V(T)$. Then, evidently, $v \in U_\delta$ for some $\delta \in \Gamma^d(T, e, \varphi)$. Since $\varphi(g) = \alpha$, we conclude that $\alpha \neq \delta$. Clearly, we also have $\alpha \neq \gamma(\delta)$ and, therefore, we infer that $E_\alpha(e,\varphi) = E_\alpha(e,\varphi_\delta)$. Moreover, we also have $\alpha_0 \notin \{\delta,\gamma(\delta)\}$ and, therefore, it follows that $P = P_{y_0}(\alpha_0,\alpha,\varphi_\delta)$. Clearly, v is the last vertex in $\preceq_{(y_0,P)}$ that belongs to $V(T_\delta)$. Since no colour from $\bar{\varphi}_\delta(v)$ is used on T_δ with respect to φ_δ, we infer that $\bar{\varphi}_\delta(v) \cap \Gamma^f(T_\delta,e,\varphi_\delta) \neq \emptyset$. Then from Lemma 3.8 follows that $v \notin F(T_\delta,e,\varphi_\delta)$ and, therefore, $\alpha \notin \Gamma^d(T_\delta,e,\varphi_\delta)$. Hence, we conclude that $E(P) \cap E_G(V(T_\delta),V(G)\setminus V(T_\delta)) = \{g\}$, which implies $V(P_1) \subseteq V(T_\delta) \subseteq X$. This settles the case.

In any of the three cases we have $V(P_1) \subseteq X$, which implies $E(P) \cap E_G(X,V(G)\setminus X) = \{g\}$. Hence the proof is finished. □

Eventually, we can show that

(21) X is strongly closed with respect to φ.

Proof of (21): Suppose, on the contrary, that X is not strongly closed with respect to φ. Since, by (19), X is closed with respect to φ, this implies that there is a colour α satisfying $\alpha \notin \bar{\varphi}(X)$ and $|E_G(X,V(G)\setminus X) \cap E_\alpha(e,\varphi)| \geq 2$. Obviously, this implies $|E_G(X,V(G)\setminus X) \cap E_\alpha(e,\varphi)| \geq 3$, because $|X|$ is odd.

If $|E_G(V(T),V(G)\setminus X) \cap E_\alpha(e,\varphi)| \geq 2$ then we would have $\alpha \in \Gamma^d(T,e,\varphi)$, but then (7) and (11) would imply $E_G(V(T),V(G)\setminus V(T)) \cap E_\alpha(e,\varphi) = \{f_\alpha^0, f_\alpha^1, f_\alpha^2\}$ and, therefore, $E_G(V(T),V(G)\setminus X) \cap E_\alpha(e,\varphi) = \{f_\alpha^0\}$, a contradiction. Consequently, we have $|E_G(V(T),V(G)\setminus X) \cap E_\alpha(e,\varphi)| \leq 1$, which leads to $|E_G(X\setminus V(T),V(G)\setminus X) \cap E_\alpha(e,\varphi)| \geq 2$.

For the path $P = P_{y_0}(\alpha_0,\alpha,\varphi)$ then follows from (20) that $|E(P) \cap E_G(X,V(G)\setminus X)| = 1$. Hence there is a colour $\delta \in \Gamma^d(T,e,\varphi)$ and an edge $f \in E_G(U_\delta,V(G)\setminus X)$ satisfying $\varphi(f) = \alpha$ and $f \notin E(P)$. Let u be the endvertex of f that belongs to $V(G)\setminus X$, and let $P' = P_u(\alpha_0,\alpha,\varphi)$. Since $f \in E(P')$ but $f \notin E(P)$, we infer that P and P' are vertex disjoint. Further we claim that $V(P') \cap V(T) = \emptyset$. To prove this, we have two consider two cases.

Case 1: $\alpha \in \Gamma^d(T,e,\varphi)$. From (7) and (11) we then conclude $E_G(V(T),V(G)\setminus V(T)) \cap E_\alpha(e,\varphi) = \{f_\alpha^0, f_\alpha^1, f_\alpha^2\}$ and, therefore, we have $|E_G(X_\alpha,V(G)\setminus X_\alpha) \cap E_\alpha(e,\varphi)| = 1$. Since $\alpha_0 \in \bar{\varphi}(X_\alpha)$ and, by (17), X_α is closed with respect to φ, we also have $E_G(X_\alpha,V(G)\setminus X_\alpha) \cap E_{\alpha_0}(e,\varphi) = \emptyset$. Since the only edge in $E_G(X_\alpha,V(G)\setminus X_\alpha) \cap E_\alpha(e,\varphi)$ must belong to $E(P)$, we conclude that $V(P') \cap X_\alpha = \emptyset$ and, therefore, also $V(P') \cap V(T) = \emptyset$.

Case 2: $\alpha \notin \Gamma^d(T,e,\varphi)$. Then the only edge in $E_G(V(T),V(G)\setminus V(T)) \cap E_\alpha(e,\varphi)$ belongs to $E(P)$. Since $\alpha_0 \in \bar{\varphi}(V(T))$ and $V(T)$ is closed with respect to φ, we conclude that $V(P') \cap V(T) = \emptyset$. This settles the case.

In any case we have $V(P') \cap V(T) = \emptyset$, which implies $E(P') \cap E(T_\delta) = \emptyset$. Moreover, from $\alpha, \alpha_0 \notin \{\delta,\gamma(\delta)\}$ we conclude that $P' = P_u(\alpha_0,\alpha,\varphi_\delta)$. Then, evidently, T_δ

is a Tashkinov tree with respect to e and $\varphi' = \varphi_\delta/P'$. From $\varphi'(f) = \alpha_0 \in \bar{\varphi}'(V(T_\delta))$ it then follows that $T' = (T_\delta, f, u)$ is also a Tashkinov tree with respect to e and φ' satisfying $|V(T')| > |V(T_\delta)| = t(G)$, a contradiction. This proves (21). $\square$

Now by (16) and (21) X is elementary and strongly closed with respect to φ. Hence Theorem 2.8 implies that G is an elementary graph, which completes the proof. $\blacksquare$

Proposition 4.2 *Let G be a critical graph with $\chi'(G) = k + 1$ for an integer $k \geq \Delta(G) + 1$. If $h(G) > t(G) - 4$ then G is an elementary graph.*

Proof: Let $h(G) > t(G) - 4$. By Proposition 3.4(a) and Lemma 3.12(a), both $t(G)$ and $h(G)$ are odd. Hence we have either $h(G) = t(G)$ or $h(G) = t(G) - 2$.

By Lemma 3.12(c), we have $\mathcal{T}^B(G) \neq \emptyset$. Hence there is an edge $e \in E_G(x, y)$, a coloring $\varphi \in \mathcal{C}_k(G - e)$ and a Tashkinov tree T with respect to e and φ satisfying $(T, e, \varphi) \in \mathcal{T}^B(G)$.

If $h(G) = t(G)$ then T consists only of its trunk, and only two colours $\alpha \in \bar{\varphi}(x)$ and $\beta \in \bar{\varphi}(y)$ are used on T with respect to φ. Therefore Proposition 3.4(f) implies that $\bar{\varphi}(v) \cap \Gamma^f(T, e, \varphi) \neq \emptyset$ for every vertex $v \in V(T)$. Then we have $\Gamma^d(T, e, \varphi) = \emptyset$, because $\Gamma^d(T, e, \varphi) \neq \emptyset$ would contradict Proposition 3.6(d). From Proposition 3.4(b) and (c) it then follows that $V(T)$ is elementary and strongly closed with respect to φ. Therefore, by Theorem 2.8, G is an elementary graph.

In the other case we have $h(G) = t(G) - 2$ and, therefore, T has the form

$$T = (y_0, e_1, y_1, \ldots, e_{p-1}, y_{p-1}, f_1, u_1, f_2, u_2)$$

where $x = y_0$, $y = y_1$ and $p = h(T)$. Clearly, exactly two colours $\alpha \in \bar{\varphi}(x)$ and $\beta \in \bar{\varphi}(y)$ are used on Ty_{p-1} with respect to φ. Moreover $\varphi(f_1) = \varphi(f_2) = \gamma \in \bar{\varphi}(\{y_0, \ldots, y_{p-1}\}) \setminus \{\alpha, \beta\}$, and for $j = 1, 2$ we have $f_j \in E_G(\{y_0, \ldots, y_{p-1}\}, u_j)$. By Proposition 3.4(b), $V(T)$ is elementary with respect to φ and therefore there is a unique vertex $y_r \in \{y_0, \ldots, y_{p-1}\}$ with $\gamma \in \bar{\varphi}(y_r)$. Since there are exactly three colours $\alpha \in \bar{\varphi}(x)$, $\beta \in \bar{\varphi}(y)$ and $\gamma \in \bar{\varphi}(y_r)$ used on T with respect to φ, we conclude from Proposition 3.4(f) that $\bar{\varphi}(v) \cap \Gamma^f(T, e, \varphi) \neq \emptyset$ for every vertex $v \in V(T) \setminus \{y_r\}$. Then Lemma 3.8 implies $F(T, e, \varphi) \subseteq \{y_r\}$. Hence (T, e, φ) fulfils the structural conditions of Lemma 4.1, and therefore G is an elementary graph. This completes the proof. $\blacksquare$

Proposition 4.3 *Let G be a critical graph with $\chi'(G) = k + 1$ for an integer $k \geq \Delta(G) + 1$. If $h(G) < 5$ then G is an elementary graph.*

Proof: Let $p = h(G) < 5$. Then, by Lemma 3.12(a), we have $p = 3$ and, by Lemma 3.12(c), there is a triple $(T, e, \varphi) \in \mathcal{T}^B(G)$. Hence T has the form

$$T = (y_0, e_1, y_1, \ldots, e_{n-1}, y_{n-1})$$

where $n = t(G)$, and $T_1 = (y_0, e_1, y_1, e_2, y_2)$ is the (α, β)-trunk of T where $\alpha \in \bar{\varphi}(y_0)$ and $\beta \in \bar{\varphi}(y_1)$. Moreover there is an edge $f \in E_G(y_2, y_0)$ with $\varphi(f) = \beta$.

Suppose that G is not an elementary graph. Then, by Proposition 4.2, we have $n = t(G) \geq p+4 = 7$. Hence there is a colour $\gamma = \varphi(e_3) = \varphi(e_4) \in \bar{\varphi}(y_j)$ for some $j \in \{1, 2, 3\}$. Without loss of generality we may assume $j = 0$, otherwise we could replace the triple (T, e, φ) by the balanced triple $(T, e, \varphi)(y_0 \to y_j)$. Therefore we have $e_3, e_4 \in E_G(\{y_1, y_2\}, \{y_3, y_4\})$ and, moreover, (y_0, y_1) is a (γ, β)-pair with respect to φ. From Theorem 3.3(e) we then conclude that there is a (γ, β)-chain P with respect to φ having endvertices y_1 and y_0 and satisfying $V(P) \subseteq V(T)$. Evidently, $p' = |V(P)|$ is odd, $f, e_3, e_4 \in E(P)$ and $y_0, y_1, y_2, y_3, y_4 \in V(P)$. Therefore we have $p' \geq 5$ and, by Lemma 3.11, there is a Tashkinov tree T' with respect to e and φ satisfying $(T', e, \varphi) \in \mathcal{T}^N(G)$ and $h(T') = p' \geq 5 > h(G)$, a contradiction. Hence G is an elementary graph. ■

Lemma 4.4 *Let G be a critical graph with $\chi'(G) = k + 1$ for $k \geq \Delta(G) + 1$. Furthermore let $h(G) = 5$, $(T, e, \varphi) \in \mathcal{T}^N(G)$ and $T' = (y_0, e_1, y_1, e_2, y_2, e_3, y_3, e_4, y_4)$ be the (α, β)-trunk of T. If $\gamma \in \bar{\varphi}(y_0)$ is a colour satisfying $E_G(V(T'), V(T) \backslash V(T')) \cap E_\gamma(e, \varphi) \neq \emptyset$, then the following statements hold:*

(1) *There are three edges $f_1 \in E_G(y_1, V(T) \backslash V(T'))$, $f_2 \in E_G(y_4, V(T) \backslash V(T'))$ and $f_3 \in E_G(y_2, y_3)$ with $\varphi(f_1) = \varphi(f_2) = \varphi(f_3) = \gamma$.*

(2) *For the two endvertices $v_1, v_2 \in V(T) \setminus V(T')$ of the two edges f_1, f_2 we have $E_G(v_1, v_2) \cap E_\alpha \neq \emptyset$ and $E_G(v_1, v_2) \cap E_\beta \neq \emptyset$.*

Proof: By definition we have $e_1 = e$, $\varphi(e_2) = \varphi(e_4) = \alpha \in \bar{\varphi}(y_0)$, $\varphi(e_3) = \beta \in \bar{\varphi}(y_1)$ and there is an edge $e_0 \in E_G(y_4, y_0)$ with $\varphi(e_0) = \beta$.

Let $P_1 = P_{y_0}(\gamma, \beta, \varphi)$. By Theorem 3.3(e), P_1 is a path of even length having endvertices y_0 and y_1. Then, by Lemma 3.11, there is a Tashkinov tree T_1 with respect to e and φ satisfying $(T_1, e, \varphi) \in \mathcal{T}^N(G)$ and $h(T_1) = |V(P_1)|$. Since $h(G) = 5$, we conclude that $|V(P_1)| \leq 5$. Since y_0, y_1 are the endvertices of P_1 and $\varphi(e_0) = \beta$, we have $e_0 \in E(P_1)$ and $y_0, y_1, y_4 \in V(P_1)$.

Now we claim that $E_G(\{y_2, y_3\}, V(T) \backslash V(T')) = \emptyset$. Suppose this is not true. Then there is an edge $g \in E_G(\{y_2, y_3\}, V(T) \backslash V(T'))$ with $\varphi(g) = \gamma$. Let $v \in V(T) \backslash V(T')$ be the second endvertex of g. We conclude that none of the three vertices y_2, y_3, v belongs to $V(P_1)$, otherwise all three would belong to $V(P_1)$ and therefore we would have $|V(P_1)| \geq 6$, a contradiction. Hence, by Theorem 3.3(e), T is a Tashkinov tree with respect to e and $\varphi_1 = \varphi/P_1$ satisfying $\alpha \in \bar{\varphi}_1(y_0)$, $\gamma \in \bar{\varphi}_1(y_1)$, $\varphi_1(e_0) = \varphi_1(g) = \gamma$ and $\varphi_1(e_2) = \varphi_1(e_4) = \alpha$. Evidently, $P_2 = P_{y_0}(\alpha, \gamma_1, \varphi_1)$ contains the two subpaths $P(y_0, e_0, y_4, e_4, y_3)$ and $P(y_1, e_2, y_2)$, which implies $g \in E(P_2)$ and $v \in V(P_2)$. Hence we have $|V(P_2)| \geq 6$ and, by Lemma 3.11, there is a Tashkinov tree T_2 with respect to e and φ_1 satisfying $(T_2, e, \varphi_1) \in \mathcal{T}^N(G)$ and $h(T_2) = |V(P_2)| \geq 6 > 5 = h(G)$, a contradiction. This proves the claim.

Since $\gamma \in \bar{\varphi}(y_0)$ and since, by Proposition 3.4(b), $V(T)$ is elementary with respect to φ, there are edges $f_3 \in E_G(y_2, V(T'))$ and $f_3' \in E_G(y_3, V(T'))$ with $\varphi(f_3) = \varphi(f_3') = \gamma$. Then $f_3 \neq f_3'$ would imply $f_3, f_3' \in E_G(\{y_2, y_3\}, \{y_1, y_4\})$ and therefore $E_G(V(T'), V(T) \setminus V(T')) \cap E_\gamma(e, \varphi) = \emptyset$, a contradiction. Consequently, we have $f_3 = f_3' \in E_G(y_2, y_3)$ with $\varphi(f_3) = \gamma$, exactly as claimed in (1).

Since $V(T)$ is elementary with respect to φ, there are edges $f_1 \in E_G(y_1)$ and $f_2 \in E_G(y_4)$ with $\varphi(f_1) = \varphi(f_2) = \gamma$. If $f_1 = f_2$ then we would have $E_G(V(T'), V(T) \setminus V(T')) \cap E_\gamma(e, \varphi) = \emptyset$, a contradiction. Consequently, we have $f_1 \neq f_2$ and, therefore, $f_1 \in E_G(y_1, V(T) \setminus V(T'))$ and $f_2 \in E_G(y_4, V(T) \setminus V(T'))$, which, eventually, proves (1).

Let v_1, v_2 the two endvertices of f_1, f_2 belonging to $V(T) \setminus V(T')$. Then P_1 contains the two subpaths $P(y_0, e_0, y_4, f_2, v_2)$ and $P(y_1, f_1, v_1)$. Since $V(P_1) \leq 5$, there is an edge $g_1 \in E_G(v_1, v_2)$ with $\varphi(g_1) = \beta$ and, therefore, we have $P_1 = P(y_1, e_5, y_5, g_1, y_6, e_6, y_4, e_0, y_0)$, which proves part of (2).

Let $P_3 = P_{y_0}(\alpha, \beta, \varphi) = P(y_1, e_2, y_2, e_3, y_3, e_4, y_4, e_0, y_0)$ and $\varphi_2 = \varphi / P_3$. Then, by Theorem 3.3(e), T is a Tashkinov tree with respect to e and φ_2, satisfying $\alpha \in \bar{\varphi}_2(y_1)$, $\gamma \in \bar{\varphi}_2(y_0)$, $\varphi_2(e_0) = \alpha$ and $\varphi_2(f_1) = \varphi_2(f_2) = \gamma$. Hence, $P_4 = P_{y_0}(\gamma, \alpha, \varphi_2)$ contains the two subpaths $P(y_0, e_0, y_4, f_2, v_2)$ and $P(y_1, f_1, v_1)$ and, therefore, we have $|V(P_4)| \geq 5$. If we had $|V(P_4)| > 5$ then, by Lemma 3.11, there would be a Tashkinov tree T_3 with respect to e and φ_2 satisfying $(T_3, e, \varphi_2) \in \mathcal{T}^N(G)$ and $h(T_3) = |V(P_4)| > 5 = h(G)$, a contradiction. Hence we have $|V(P_4)| = 5$ and there is an edge $g_2 \in E_G(v_1, v_2)$ with $\varphi_2(g_2) = \alpha$. Since $g_2 \notin E(P_3)$, we also have $\varphi(g_2) = \alpha$, which eventually proves (2). ■

Proposition 4.5 *Let G be a critical graph with $\chi'(G) = k + 1$ for an integer $k \geq \Delta(G) + 1$. If $t(G) < 11$ then G is an elementary graph.*

Proof: Suppose, on the contrary, that $t(G) < 11$ but G is not elementary. By Proposition 3.4(a), $t(G)$ is odd and, therefore, we have $t(G) \leq 9$. Moreover, from Proposition 4.2 and Proposition 4.3 we conclude that $5 \leq h(G) \leq t(G) - 4 \leq 5$ and, therefore, we have $h(G) = 5$ and $t(G) = 9$.

Let $(T, e, \varphi) \in \mathcal{T}^B(G)$. Then T has the form

$$T = (y_0, e_1, y_1, e_2, y_2, e_3, y_3, e_4, y_4, e_5, y_5, e_6, y_6, e_7, y_7, e_8, y_8)$$

where $e_1 = e$, $\varphi(e_2) = \varphi(e_4) = \alpha \in \bar{\varphi}(y_0)$, $\varphi(e_3) = \beta \in \bar{\varphi}(y_1)$, $\varphi(e_5) = \varphi(e_6) = \gamma_1 \in \bar{\varphi}(\{y_0, \ldots, y_4\})$ and $\varphi(e_7) = \varphi(e_8) = \gamma_2 \in \bar{\varphi}(\{y_0, \ldots, y_6\})$. Moreover, $T_1 = Ty_4$ is the (α, β)-trunk of T and there is an edge $e_0 \in E_G(y_4, y_0)$ with $\varphi(e_0) = \beta$.

Clearly, $\gamma_1 \in \bar{\varphi}(y_i)$ for some $i \in \{0, \ldots, 4\}$. Without loss of generality we may assume $i = 0$, otherwise we could replace the triple (T, e, φ) by the balanced triple

$(T,e,\varphi)(y_0 \to y_i)$. Since $e_5 \in E_G(\{y_0,\dots,y_4\},y_5)$, we conclude from Lemma 4.4 that there are five edges f_1,f_2,f_3,g_1,g_2 satisfying $f_1 \in E_G(y_1,v_1)$ for a vertex $v_1 \in \{y_5,\dots,y_8\})$, $f_2 \in E_G(y_4,v_2)$ for a vertex $v_2 \in \{y_5,\dots,y_8\})$, $\varphi(f_1)=\varphi(f_2)=\gamma_1$, $f_3 \in E_G(y_2,y_3)$, $\varphi(f_3)=\gamma_1$, $g_1,g_2 \in E_G(v_1,v_2)$, $\varphi(g_1)=\alpha$ and $\varphi(g_2)=\beta$. In particular, this implies $\{e_5,e_6\}=\{f_1,f_2\}$ and $\{y_5,y_6\}=\{v_1,v_2\}$.

Now we have $\varphi(e_2)=\varphi(e_4)=\varphi(g_1)=\alpha \in \bar\varphi(y_0)$, $\varphi(e_0)=\varphi(e_3)=\varphi(g_2)=\beta \in \bar\varphi(y_1)$, $\varphi(f_1)=\varphi(f_2)=\varphi(f_3)=\gamma_1 \in \bar\varphi(y_0)$ and $e_7,e_8 \in E_G(\{y_0,\dots,y_6\},\{y_7,y_8\})$ and, therefore, $\gamma_2 \notin \{\alpha,\beta,\gamma_1\}$. Since, by Proposition 3.4(b), $V(T)$ is elementary and closed with respect to φ, there are three edges $f_4,g_3,g_4 \in E_G(y_7,y_8)$ satisfying $\varphi(f_4)=\gamma_1$, $\varphi(g_3)=\alpha$ and $\varphi(g_4)=\beta$.

We may assume that $\gamma_2 \in \bar\varphi(\{y_0,\dots,y_4\})$, otherwise we could replace T by $T_1 = (y_0,e_1,y_1,f_1,v_1,g_2,v_2,f_2,y_4,e_2,y_2,e_4,y_3,e_7,y_7,e_8,y_8)$. Obviously $(T_1,e,\varphi) \in \mathcal{T}^B(G)$ and T_1y_4 is the (γ_1,β)-trunk of T. Hence, T_1 has the same structure as T, just the two colours α and γ_1 changed their role.

Now we claim that $E_G(\{y_0,\dots,y_4\},\{y_7,y_8\}) \cap E_{\gamma_2}(e,\varphi) \neq \emptyset$. Suppose this is not true. Then we have $e_7,e_8 \in E_G(\{y_5,y_6\},\{y_7,y_8\})$ and, by symmetry, we may assume $e_7 \in E_G(y_5,y_7)$ and $e_8 \in E_G(y_6,y_8)$. Evidently, the chain $P_1 = P_{y_7}(\gamma_2,\beta,\varphi) = P(y_7,e_7,y_5,g_2,y_6,e_8,y_8,g_4,y_7)$ is a cycle and T is a Tashkinov tree with respect to e and $\varphi_1 = \varphi/P_1$. Moreover, we have $P_2 = P_{y_0}(\gamma_1,\beta,\varphi_1) = P(y_1,f_1,v_1,e_7,y_7,f_4,y_8,e_8,v_2,f_2,y_4,e_0,y_0)$, and therefore, by Lemma 3.11, there is a Tashkinov tree T_2 with respect to e and φ_1 satisfying $(T_2,e,\varphi_1) \in \mathcal{T}^N(G)$ and $h(T_2)=|V(P_2)|=7>h(G)$, a contradiction. This proves the claim.

Since we have $\gamma_2 \in \bar\varphi(y_j)$ for some $j \in \{0,\dots,4\}$, we can construct a new Tashkinov tree as follows. In the case $j=0$ let $(T',e',\varphi')=(T,e,\varphi)$, otherwise let $(T',e',\varphi')=(T,e,\varphi)(y_0 \to y_j)$. In any case we have $(T',e',\varphi') \in \mathcal{T}^B(G)$, and T' has the form

$$T' = (y'_1,e'_1,y'_2,e'_3,y'_3,e'_4,y'_4,e_5,y_5,e_6,y_6,e_7,y_7,e_8,y_8)$$

where $\{y'_0,\dots,y'_4\}=\{y_0,\dots,y_4\}$ and $\gamma_2 \in \bar\varphi'(y'_0)$. By Lemma 4.4, there are two vertices $v'_1,v'_2 \in \{y_5,\dots,y_8\}$ and four edges f'_1,f'_2,f'_3,g'_1 satisfying $f'_1 \in E_G(y'_1,v'_1)$, $f'_2 \in E_G(y'_4,v'_2)$, $\varphi'(f_1)=\varphi'(f_2)=\gamma_2$, $f'_3 \in E_G(y'_2,y'_3)$, $\varphi'(f'_3)=\gamma_2$, $g'_1 \in E_G(v'_1,v'_2)$ and $\varphi'(g'_1)=\alpha$. Consequently, we have $f'_1,f'_2 \in E_G(\{y_0,\dots,y_4\},\{y_5,\dots,y_8\})$, $f'_3 \in E_G(\{y_0,\dots,y_4\},\{y_0,\dots,y_4\})$, $\varphi(f'_1)=\varphi(f'_2)=\varphi(f'_3)=\gamma_2$ and $\varphi(g'_1)=\alpha$. This implies $|E_G(\{y_0,\dots,y_4\},\{y_5,\dots,y_8\}) \cap E_{\gamma_2}(e,\varphi)|=2$. Since we also have $E_G(\{y_0,\dots,y_4\},\{y_7,y_8\}) \cap E_{\gamma_2}(e,\varphi) \neq \emptyset$, we conclude that $\{y_7,y_8\} \cap \{v'_1,v'_2\} \neq \emptyset$. Then from $\varphi(g'_1)=\varphi(g_3)=\alpha$ follows $\{y_7,y_8\}=\{v'_1,v'_2\}$.

Now we have $\gamma_1 \in \bar\varphi(y_0)$, $\gamma_2 \in \bar\varphi(y_j)$ for some $j \in \{0,\dots,4\}$. Moreover, Proposition 3.4(f) implies $|\bar\varphi(v) \setminus \{\alpha,\beta\}| \geq 1$ for every $v \in V(T)$. Since no colours beside α,β,γ_1 and γ_2 are used on T with respect to φ, we conclude that for every vertex $v \in V(T) \setminus \{y_0,y_j\}$ the set $\bar\varphi(v)$ contains at least one free colour with respect to (T,e,φ). Then from Lemma 3.8 follows that $F(T,e,\varphi) \subseteq \{y_0,y_j\}$. Since $\gamma_1 \neq \gamma_2$

and $e_5, e_6, e_7, e_8 \in E_G(\{y_0, \ldots, y_4\})$, the triple (T, e, φ) fulfils the structural conditions from Lemma 4.1 and, therefore, G is an elementary graph. This completes the proof. ■

Lemma 4.6 *Let G be a graph with $\chi'(G) = k+1$ for an integer $k \geq \Delta(G)+1$ and let $(T, e, \varphi) \in \mathcal{T}(G)$. Moreover, let $\alpha, \beta \in \{1, \ldots, k\}$ and let P be an (α, β)-chain with respect to φ satisfying $V(P) \cap V(T) = \emptyset$. Then for the coloring $\varphi' = \varphi/P$ the following invariants hold:*

- *$(T, e, \varphi') \in \mathcal{T}(G)$, $\Gamma^f(T, e, \varphi') = \Gamma^f(T, e, \varphi)$ and $\Gamma^d(T, e, \varphi') = \Gamma^d(T, e, \varphi)$.*
- *$D(T, e, \varphi') = D(T, e, \varphi)$.*

Proof: From $V(P) \cap V(T) = \emptyset$ we conclude that $\varphi'(f) = \varphi(f)$ for every edge $f \in E_{G-e}(V(T), V(G))$. Evidently, this implies $(T, e, \varphi') \in \mathcal{T}(G)$, $\Gamma^f(T, e, \varphi') = \Gamma^f(T, e, \varphi)$ and $\Gamma^d(T, e, \varphi') = \Gamma^d(T, e, \varphi)$.

Now let $v \in D(T, e, \varphi)$. Then there are two colours $\gamma \in \Gamma^f(T, e, \varphi)$ and $\delta \in \Gamma^d(T, e, \varphi)$ such that v is the first vertex in the linear order $\preceq_{(u, P_1)}$ that belongs to $V(G) \backslash V(T)$, where $u \in V(T)$ is the unique vertex with $\gamma \in \bar{\varphi}(u)$ and $P_1 = P_u(\gamma, \delta, \varphi)$. Consequently, for $P_2 = uP_1v$ we have $E(P_2) \subseteq E_{G-e}(V(T))$. Since $\gamma \in \Gamma^f(T, e, \varphi')$, $\delta \in \Gamma^d(T, e, \varphi')$ and $\gamma \in \bar{\varphi}'(u)$, we conclude that $\varphi'(f) = \varphi(f)$ for every edge $f \in E(P_2)$. Hence, we have $P_2 = uP_1'v$ where $P_1' = P_u(\gamma, \delta, \varphi')$. This leads to $v \in D(T, e, \varphi')$, and therefore we have $D(T, e, \varphi) \subseteq D(T, e, \varphi')$.

Since P is also an (α, β)-chain with respect to φ' and since we not only have $\varphi' = \varphi/P$ but also $\varphi = \varphi'/P$, we conclude $D(T, e, \varphi') \subseteq D(T, e, \varphi)$ in an analogous way. Consequently $D(T, e, \varphi') = D(T, e, \varphi)$, and the proof is finished. ■

Lemma 4.7 *Let G be a graph with*

$$\chi'(G) > \frac{m}{m-1}\Delta(G) + \frac{m-3}{m-1}$$

for an odd integer $m \geq 3$. Moreover, let $(T, e, \varphi) \in \mathcal{T}(G)$ and $Z = V(T) \cup D(T, e, \varphi)$. Then the following statements hold:

(a) *$|Z| \leq m-2$.*

(b) *If $|Z| = m-2$ then G is elementary.*

Proof: Since $\chi'(G) > \Delta(G)$, we have $\Delta(G) \geq 2$. Then $\chi'(G) > \frac{m}{m-1}\Delta(G) + \frac{m-3}{m-1} \geq \Delta(G)+1$ and therefore $\chi'(G) \geq \Delta(G)+2$. Hence, for $k = \chi'(G)-1$ we have $k \geq \Delta(G)+1$ and $\varphi \in \mathcal{C}_k(G-e)$.

From Proposition 3.5 and Proposition 3.7 we conclude that Z is elementary with respect to φ. Then Proposition 2.7(c) implies $|Z| \leq m-1$.

Now suppose $|Z| = m-1$. Since we have $k \geq \Delta(G)$, there is a colour $\alpha \in \bar{\varphi}(Z)$. Moreover, Z is elementary with respect to φ and $|Z|$ is even, so there is an edge $g \in E_G(Z, V(G) \setminus Z)$ having an endvertex $z \in V(G) \setminus Z$ and satisfying $\varphi(g) = \alpha$. Therefore $F = (g, z)$ is a fan at Z with respect to φ, and Theorem 3.10 implies that $Z \cup \{z\}$ is elementary with respect to φ. Since $|Z \cup \{z\}| = m$, this contradicts Proposition 2.7(c). Consequently, we have $|Z| \leq m-2$ and (a) is proved.

For the proof of (b) let $|Z| = m-2$. We claim that Z is closed with respect to φ. Suppose this is not true. Then there is a colour $\alpha' \in \bar{\varphi}(Z)$ satisfying $E_G(Z, V(G) \setminus Z) \cap E_{\alpha'}(e, \varphi) \neq \emptyset$. Since Z is elementary with respect to φ and $|Z|$ is odd, we conclude that there are at least two distinct edges $g_1, g_2 \in E_G(Z, V(G) \setminus Z)$ having endvertices $z_1, z_2 \in V(G) \setminus Z$ and satisfying $\varphi(g_1) = \varphi(g_2) = \alpha'$. Clearly, $z_1 \neq z_2$ and, therefore, $F' = (g_1, z_1, g_2, z_2)$ is a fan at Z with respect to φ. Hence, by Theorem 3.10, $Z \cup \{z_1, z_2\}$ is elementary with respect to φ, but then $|Z \cup \{z_1, z_2\}| = m$ contradicts Proposition 2.7(c). This proves the claim that Z is closed with respect to φ.

Now we want to show that Z is also strongly closed with respect to φ. Suppose this is not true. Then there is a colour $\delta \in \{1, \ldots, k\}$ satisfying $\delta \notin \bar{\varphi}(Z)$ and $|E_G(Z, V(G) \setminus Z) \cap E_\delta(e, \varphi)| \geq 2$. Since $|Z|$ is odd, we conclude that $|E_G(Z, V(G) \setminus Z) \cap E_\delta(e, \varphi)| \geq 3$. Moreover, by Proposition 3.4(g), there is a colour $\gamma \in \Gamma^f(T, e, \varphi)$, and there is a unique vertex $v \in V(T)$ satisfying $\gamma \in \bar{\varphi}(v)$. Let $P = P_v(\gamma, \delta, \varphi)$. Then P is a path and v is an endvertex of P. Since Z is elementary with respect to φ and $\delta \notin \bar{\varphi}(Z)$, the other endvertex of P belongs to $V(G) \setminus Z$. Hence in the linear order $\preceq_{(v,P)}$ there is a first vertex u that belongs to $V(G) \setminus Z$. We claim that there is a coloring $\varphi' \in \mathcal{C}_k(G-e)$ satisfying $(T, e, \varphi') \in \mathcal{T}(G)$, $D(T, e, \varphi') = D$, $E_\delta(e, \varphi') = E_\delta(e, \varphi)$ and $P_v(\gamma, \delta, \varphi') = vPu$. For the proof of this claim we have to consider the following two cases:

Case 1: $\bar{\varphi}(u) \cap \bar{\varphi}(Z) \neq \emptyset$. Then, for some colour $\beta \in \bar{\varphi}(u) \cap \bar{\varphi}(Z)$, let $P_1 = P_u(\gamma, \beta, \varphi)$. Since Z is closed with respect to φ, we conclude that $V(P_1) \cap Z = \emptyset$. By Lemma 4.6, for the coloring $\varphi' = \varphi / P_1$ we have $(T, e, \varphi') \in \mathcal{T}(G)$ and $D(T, e, \varphi') = D$. Moreover, $\gamma \in \bar{\varphi}'(u)$ implies $P_v(\gamma, \delta, \varphi') = vPu$. Hence, φ' has the desired properties.

Case 2: $\bar{\varphi}(u) \cap \bar{\varphi}(Z) = \emptyset$, i.e. $Z \cup \{u\}$ is elementary with respect to φ. Then there is a vertex $u' \in V(G) \setminus Z$ and an edge $f \in E_G(u, u')$ with $\varphi(f) = \gamma$. From Proposition 2.7(c) we infer that $Z \cup \{u, u'\}$ is not elementary with respect to φ. Since $Z \cup \{u\}$ is elementary with respect to φ, this implies that $\bar{\varphi}(u') \cap \bar{\varphi}(Z \cup \{u\}) \neq \emptyset$. We have to consider three subcases.

Case 2a: There is a colour $\gamma_1 \in \bar{\varphi}(u') \cap \bar{\varphi}(u)$. Then we can simply obtain the desired coloring φ' from φ by recolouring the edge f with the colour γ_1.

Case 2b: There is colour $\gamma_1 \in \bar{\varphi}(u') \cap \bar{\varphi}(Z)$ satisfying $\gamma_1 \in \Gamma^f(T, e, \varphi)$. Then there is a unique vertex $v' \in V(T)$ with $\gamma_1 \in \bar{\varphi}(v')$. Moreover, let $\gamma_2 \in \bar{\varphi}(u)$. Since $Z \cup \{u\}$ is elementary with respect to φ, we clearly have $\gamma_1 \neq \gamma_2$. Now let $P_2 = P_{v'}(\gamma_1, \gamma_2, \varphi)$. Then P_2 is a path, where v' is an endvertex of P_2. Moreover, we have $E_{\gamma_1}(T, e, \varphi) \subseteq E(P_2)$, which follows either from $|E_{\gamma_2}(T, e, \varphi)| = 1$ if this is the case, or otherwise from Proposition 3.6(b).

If u is the second endvertex of P_2 then, evidently, u' is not an endvertex of P_2 and, therefore, u' doesn't belong to $V(P_2)$ at all. For $P_3 = P_{u'}(\gamma_1, \gamma_2, \varphi)$ we then conclude that $V(P_3) \cap V(T) = \emptyset$ and $u \notin V(P_3)$. For the coloring $\varphi_3 = \varphi/P_3$ we then have $\gamma_2 \in \bar{\varphi}_2(u) \cap \bar{\varphi}_2(u')$ and since, by Lemma 4.6, we still have $(T, e, \varphi_3) \in \mathcal{T}(G)$ and $D(T, e, \varphi_3) = D$, we can obtain the desired coloring φ' from φ_3 by recolouring the edge f using the colour γ_2.

If otherwise u is not the second endvertex of P_3 then, evidently, u doesn't belong to $V(P_3)$ at all. For $P_4 = P_u(\gamma_1, \gamma_2, \varphi)$ we then conclude that $V(P_4) \cap V(T) = \emptyset$. For the coloring $\varphi_4 = \varphi/P_4$ we then have $\gamma_1 \in \bar{\varphi}_4(u) \cap \bar{\varphi}_4(v')$ and, since, by Lemma 4.6, we still have $(T, e, \varphi_4) \in \mathcal{T}(G)$ and $D(T, e, \varphi_4) = D$, we can obtain the desired coloring φ' from φ_4 by recolouring analogously to Case 1.

Case 2c: There is colour $\gamma_1 \in \bar{\varphi}(u') \cap \bar{\varphi}(Z)$ satisfying $\gamma_1 \notin \Gamma^f(T, e, \varphi)$. Then let $\gamma_3 \in \Gamma^f(T, e, \varphi)$. Evidently, $\gamma_1 \neq \gamma_3$ and $P_5 = P_{u'}(\gamma_1, \gamma_3, \varphi)$ is a path with $V(P_5) \cap Z = \emptyset$, because Z is closed with respect to φ. Therefore, by Lemma 4.6, the coloring $\varphi_5 = \varphi/P_5$ satisfies $(T, e, \varphi_5) \in \mathcal{T}(G)$, $\Gamma^f(T, e, \varphi_5) = \Gamma^f(T, e, \varphi)$, $\Gamma^d(T, e, \varphi_5) = \Gamma^d(T, e, \varphi)$ and $D(T, e, \varphi_5) = D$. Moreover, we have $\gamma_3 \in \bar{\varphi}_5(u') \cap \Gamma^f(T, e, \varphi_5)$. Hence, we can obtain the desired coloring φ' from φ_5 by recolouring analogously to Case 2b.

Hence the claim is proved and there is a coloring $\varphi' \in \mathcal{C}_k(G - e)$ satisfying $(T, e, \varphi') \in \mathcal{T}(G)$, $D(T, e, \varphi') = D$, $E_\delta(e, \varphi') = E_\delta(e, \varphi)$ and $P' = P_v(\gamma, \delta, \varphi') = vPu$. Evidently, we have $|E(P') \cap E_G(Z, V(G) \setminus Z)| = 1$. Since, by assumption, we also have $|E_G(Z, V(G) \setminus Z) \cap E_\delta(e, \varphi)| \geq 3$, there must be two edges $f_1, f_2 \in E_G(Z, V(G) \setminus Z) \setminus E(P')$ having endvertices $v_1, v_2 \in Z$ and $u_1, u_2 \in V(G) \setminus Z$ and satisfying $\varphi(f_1) = \varphi(f_2) = \delta$, which also implies $\varphi'(f_1) = \varphi'(f_2) = \delta$.

Now let $P'_1 = P_{u_1}(\gamma, \delta, \varphi')$ and $P'_2 = P_{u_2}(\gamma, \delta, \varphi')$. Note that P'_1 and P'_2 may be equal. Since $P' \cap \{u_1, u_2\} = \emptyset$, both chains P'_1 and P'_2 are vertex disjoint to P'. Moreover, Proposition 3.6(b) implies that $V(P'_1) \cap V(T) = \emptyset$ and $V(P'_2) \cap V(T) = \emptyset$. If $P'_1 = P'_2$ then let $\varphi'_2 = \varphi'/P'_1$, otherwise let $\varphi'_2 = (\varphi'/P'_1)/P'_2$. Then from Lemma 4.6 we conclude $(T, e, \varphi'_2) \in \mathcal{T}(G)$, $\Gamma^f(T, e, \varphi'_2) = \Gamma^f(T, e, \varphi') = \Gamma^f(T, e, \varphi)$, $\Gamma^d(T, e, \varphi'_2) = \Gamma^d(T, e, \varphi') = \Gamma^d(T, e, \varphi)$ and $D(T, e, \varphi'_2) = D$. Moreover, we have $\varphi'_2(f_1) = \varphi'_2(f_2) = \gamma$ and, therefore, $F = (f_1, u_1, f_2, u_2)$ is a fan at Z with respect to φ'_2. From Theorem 3.10 then follows that $Z \cup \{u_1, u_2\}$ is elementary with respect to φ'_2, but since $|Z \cup \{u_1, u_2\}| = m$, this contradicts Proposition 2.7(c). Consequently, Z is strongly closed with respect to φ and, by Theorem 2.8, G is an elementary graph. This completes the proof. ■

Proposition 4.8 *Let G be a critical graph with*

$$\chi'(G) > \frac{m}{m-1}\Delta(G) + \frac{m-3}{m-1}$$

for an odd integer $m \geq 3$. Then the following statements hold:

(a) *If $t(G) > m-4$ then G is elementary.*

(b) *If $t(G) = m-4$ and $h(G) > t(G) - 8$ then G is elementary.*

Proof: Let $t(G) > m-4$. Since m is odd and, by Proposition 3.4(a), $t(G)$ is odd, too. This implies $t(G) \geq m-2$. Evidently, for any $(T, e, \varphi) \in \mathcal{T}(G)$ we have $|V(T) \cup D(T, e, \varphi)| \geq m-2$. Hence Lemma 4.7 implies that $|V(T) \cup D(T, e, \varphi)| = m-2$ and G is elementary. This proves (a).

Now let $t(G) = m-4$ and $h(T) > |V(T)| - 8$. By Lemma 3.12 there is a triple $(T, e, \varphi) \in \mathcal{T}^B(G)$. Therefore, we have $e \in E_G(x, y)$ for two vertices $x, y \in V(T)$ and $\varphi \in \mathcal{C}^k(G-e)$ where $k = \chi'(G)-1$. Since $\chi'(G) > \Delta(G)$, we conclude that $\Delta(G) \geq 2$. Then $\chi'(G) > \frac{m}{m-1}\Delta(G) + \frac{m-3}{m-1} \geq \Delta(G) + 1$ and, therefore, $\chi'(G) \geq \Delta(G) + 2$ and $k \geq \Delta(G) + 1$.

If $\Gamma^d(T, e, \varphi) = \emptyset$ then Proposition 3.4 implies that $V(T)$ is elementary as well as strongly closed with respect to φ, and therefore, by Theorem 2.8, G is elementary. So for the rest of the proof we assume $\Gamma^d(T, e, \varphi) \neq \emptyset$. In particular, this implies $D = D(T, e, \varphi) \neq \emptyset$. If $|D| \geq 2$ then $|V(T) \cup D| \geq m-2$ and from Lemma 4.7 follows that $|V(T) \cup D| = m-2$ and G is elementary. So from now on we may assume that $|D| = 1$.

Let $\delta \in \Gamma^d(T, e, \varphi)$ and $E' = E_\delta(T, e, \varphi)$. Then, by Proposition 3.4(e), we have $|E'| \geq 3$. Let s be the number of vertices $v \in V(T)$, such that $\bar{\varphi}(v)$ contains no free colour with respect to φ. Then, obviously, we have $s \geq |E'| - |D| \geq |E'| - 1 \geq 2$.

Let $\alpha_1 \in \bar{\varphi}(x)$ and $\alpha_2 \in \bar{\varphi}(y)$ be the two colours used on the trunk of T with respect to φ. Clearly, we have $|\bar{\varphi}(v)| \geq k - \Delta(G) + 1$ for $v \in \{x, y\}$ and $|\bar{\varphi}(v)| \geq k - \Delta(G)$ for $v \in V(T) \setminus \{x, y\}$. This implies that $|\bar{\varphi}(v) \setminus \{\alpha_1, \alpha_2\}| \geq k - \Delta(G)$ for every $v \in V(T)$. Since (T, e, φ) is a balanced triple and $h(T) > |V(T)| - 8$, we conclude that beside α_1, α_2 there are at most 3 other colours used on T with respect to φ, which leads to $s \leq \frac{3}{k-\Delta(G)}$.

Now we have $2 \leq |E'| - 1 \leq s \leq \frac{3}{k-\Delta(G)}$ which implies $k = \Delta(G) + 1$ and, therefore, $2 \leq |E'| - 1 \leq 3$. Since, by Proposition 3.4(e), $|E'|$ is odd, we then conclude that $|E'| = 3$.

Let $E' = \{f_1, f_2, f_3\}$, and for $i = 1, 2, 3$ let u_i be the endvertex of f_i belonging to $V(G) \setminus V(T)$. Clearly, one of these vertices belongs to D, say $u_1 \in D$. Since $|D| = 1$, we then have $D = \{u_1\}$. From Proposition 3.5 and Proposition 3.7 we conclude that $Z = V(G) \cup D = V(T) \cup \{u_1\}$ is elementary with respect to φ. Since, by Proposition 3.4, $V(T)$ is closed with respect to φ, there is a vertex $u \in V(G) \backslash Z$ and an edge $f \in E_G(u_1, u)$ with $\varphi(f) \in \bar{\varphi}(V(G))$. Then (f, u) is a fan at Z with respect to φ, and therefore, by Theorem 3.10, $X = Z \cup \{u\}$ is elementary with respect to φ.

Now we claim that $\delta \notin \bar{\varphi}(X)$. Suppose this is not true. Since $\delta \notin \bar{\varphi}(Z)$, this implies that $\delta \in \bar{\varphi}(u)$. Then $(f, u, f_2, u_2, f_3, u_3)$ is a fan at Z with respect to φ, and therefore, by Theorem 3.10, $X_1 = X \cup \{u_2, u_3\}$ is elementary with respect to φ. Since $|X_1| = m$, this contradicts Proposition 2.7(c). This proves the claim.

This implies $k \geq |\bar{\varphi}(X)| + 1$. Since $|\bar{\varphi}(v)| \geq 2$ for $v \in \{x, y\}$ and $|\bar{\varphi}(v)| \geq 1$ for $v \in V(T) - \{x, y\}$ and since X is elementary with respect to φ, we have $|\bar{\varphi}(X)| \geq |X| + 2 = m$. Hence on the one hand we have $k \geq m + 1$. On the other hand we have $k + 1 = \chi'(G) > \frac{m}{m-1}\Delta(G) + \frac{m-3}{m-1} = k + \frac{k-3}{m-1}$, which leads to $k < m + 2$. Since k and m both are integers, we conclude that $k = m + 1 = |\bar{\varphi}(X)| + 1$.

Now we claim that X is closed with respect to φ. Suppose this is not true. Then there is a colour $\alpha \in \bar{\varphi}(X)$ satisfying $E_1 = E_G(X, V(G)\backslash X) \cap E_\alpha(e, \varphi) \neq \emptyset$. Since X is elementary with respect to φ, there is a unique vertex in X where the colour α is missing with respect to φ. Moreover, $|X| = m - 2$ is odd and, therefore, $|E_1|$ is even and $|E_1| \geq 2$. Hence, there is at least one edge $f' \in E_1$ having an endvertex in Z. Let $u' \in V(G)\backslash X$ be the other endvertex of f'. Then (f, u, f', u') is a fan at Z with respect to φ and, by Theorem 3.10, $X_2 = X \cup \{u'\}$ is elementary with respect to φ. From $k = m + 1$ and $|X_2| = m - 1$ we conclude $k = |X_2| + 2 \leq |\bar{\varphi}(X_2)| \leq k$ and, therefore, $|\bar{\varphi}(X_2)| = k$. This implies $\delta \in \bar{\varphi}(X_2)$, and from $\delta \notin \bar{\varphi}(X)$ we then infer $\delta \in \bar{\varphi}(u')$. Consequently, we have $u' \notin \{u_2, u_3\}$ and, therefore, at least one of the vertices u_2, u_3 doesn't belong to X_2, say $u_2 \notin X_2$. Then, evidently, (f, u, f', u', f_2, u_2) is a fan at Z with respect to φ, and therefore, by Theorem 3.10, $X_3 = X_2 \cup \{u_2\}$ is elementary with respect to φ, but $|X_3| = m$ contradicts Proposition 2.7(c). This proves the claim.

Let $E'' = E_G(X, V(G) \setminus X) \cap E_\delta(e, \varphi)$. Since $\delta \notin \bar{\varphi}(X)$ and $|X|$ is odd, we conclude that $|E''| \geq 1$ is odd, too. We claim that $|E''| = 1|$. Suppose on the contrary $|E''| > 1$. Since $|E''|$ is odd, this implies $|E''| \geq 3$. From $E' = \{f_1, f_2, f_3\}$ and $f_1 \in E_G(X, X)$ it then follows that $|E''| = 3$ and $E'' = \{f_2, f_3, g\}$ for an edge $g \in E_G(u, v)$ where $v \in V(G)\backslash X$. Let $\beta \in \bar{\varphi}(v)$. Evidently, we have $\beta \neq \delta$, and since $k = |\bar{\varphi}(X)| + 1$ and $\delta \notin \bar{\varphi}(X)$, we have $\beta \in \bar{\varphi}(X)$. Moreover, by Proposition 3.4(g), there is a colour $\gamma \in \Gamma^f(T, e, \varphi)\backslash\{\beta\}$. Now let $P = P_v(\beta, \gamma, \varphi)$ and $\varphi' = \varphi/P$. Since X is closed with respect to φ, we conclude that $V(P) \cap X = \emptyset$. By Lemma 4.6 we then have $(T, e, \varphi') \in \mathcal{T}(G)$, $\gamma \in \Gamma^f(T, e, \varphi')$, $\delta \in \Gamma^d(T, e, \varphi')$ and $D(T, e, \varphi') = \{u_1\}$. Moreover, we have $\varphi'(f_1) = \varphi'(f_2) = \varphi'(f_3) = \varphi'(g) = \delta$ and $\gamma \in \bar{\varphi}'(v)$. Since $V(T)$ and X both are closed with respect to φ, there is an edge $g' \in E_G(u_1, u)$ satisfying $\varphi'(g') = \varphi(g') = \gamma$. Let $v' \in V(T)$ be the unique vertex with $\gamma \in \bar{\varphi}'(v)$, and let $P' = P_{v'}(\gamma, \delta, \varphi')$. Since $D(T, e, \varphi') = \{u_1\}$, we conclude that u_1 is the first vertex in the linear order $\preceq_{(P',v')}$ that belongs to $V(G)\setminus V(T)$. Then, evidently, u, v are the next vertices in the linear order $\preceq_{(P',v')}$ and, moreover, v is the second endvertex of P'. Consequently, we have $f_2, f_3 \notin E(P')$, a contradiction to Proposition 3.6(b). Hence the claim is proved.

Since $k = |\bar{\varphi}(X)| + 1$ and $\delta \notin \bar{\varphi}(X)$ and since X is closed with respect to φ, we then conclude that every edge in $E_G(X, V(G)\setminus X)$ is coloured with δ with respect to φ. Hence, $|E''| = 1$ implies that X is strongly closed with respect to φ. Since X is also elementary with respect to φ, we infer from Theorem 2.8 that G is elementary. This, eventually, proves (b). ∎

In this section we have developed several sufficient conditions for a critical graph G being elementary depending on the Tashkinov-tree-related parameters $t(G)$ and $h(G)$. Eventually, using these results, we can easily prove Theorem 2.4.

Proof of Theorem 2.4: Let G be a graph with $\chi'(G) > \frac{15}{14}\Delta(G) + \frac{12}{14}$. Moreover, let H be a critical subgraph of G satisfying $\chi'(H) = \chi'(G)$. Clearly, we have $\Delta(H) \le \Delta(G)$ and, therefore,

$$\chi'(H) > \frac{15}{14}\Delta(H) + \frac{12}{14}.$$

Now we distinguish four cases.

Case 1: $t(H) < 11$. Then, by Proposition 4.5, H is elementary.

Case 2: $t(H) > 11$. Then, by Proposition 4.8(a), H is elementary.

Case 3: $t(H) = 11$ and $h(H) \le 3$. Then from Proposition 4.3 follows that H is elementary.

Case 4: $t(H) = 11$ and $h(H) > 3$. Then from Proposition 4.8(b) follows that H is elementary.

In any case H is an elementary graph, so we have $\chi'(H) = w(H)$. This implies $w(G) \le \chi'(G) = \chi'(H) = w(H) \le w(G)$ and, therefore, we have $\chi'(G) = w(G)$. Hence, G is elementary, too, and the proof is complete. ∎

5 Upper bounds for the chromatic index

From Theorem 2.4 we derived the parameter $\max\left\{\left\lfloor \frac{15}{14}\Delta + \frac{12}{14} \right\rfloor, w\right\}$ as an upper bound for the chromatic index χ'. This was stated in Corollary 2.5 already. But there are some other upper bounds for χ' that can be derived from the results of the last section. They improve some known bounds and asymptotically support Goldberg's conjecture.

For every $\epsilon > 0$ let the graph parameter τ_ϵ be defined by

$$\tau_\epsilon(G) = \max\left\{ \lfloor (1+\epsilon)\Delta(G) + 1 - 2\epsilon \rfloor\,,\ \Delta(G) - 1 + \frac{1}{2\epsilon},\ \mathcal{w}(G) \right\}.$$

Clearly, this graph parameter is monotone, i.e., $\tau_\epsilon(H) \leq \tau_\epsilon(G)$ for every subgraph H of G. The following theorem states that τ_ϵ is for every $\epsilon > 0$ an upper bound for the chromatic index χ'. In particular, this is an improvement of a result due to Sanders and Steurer [7].

Theorem 5.1 *For every $\epsilon > 0$ every graph G satisfies $\chi'(G) \leq \tau_\epsilon(G)$.*

Proof: Suppose, on the contrary, that this is not true. Then there is a minimal graph G and there is an $\epsilon > 0$ satisfying $\chi'(G) > \tau_\epsilon(G)$. Evidently, G is a critical graph, because otherwise we would have $\chi'(H) = \chi'(G) > \tau_\epsilon(G) \geq \tau_\epsilon(H)$ for a proper subgraph H of G, a contradiction to the minimality of G.

Let $\Delta = \Delta(G)$. From $\chi'(G) > \tau_\epsilon(G) \geq \mathcal{w}(G)$ we easily infer that $\Delta \geq 2$. Then we have $\tau_\epsilon(G) \geq \lfloor (1+\epsilon)\Delta + 1 - 2\epsilon \rfloor \geq \Delta + 1$ and, therefore, $\chi'(G) \geq \Delta + 2$. From Lemma 3.12 we then conclude that there is a triple $(T, e, \varphi) \in \mathcal{T}^B(G)$. Then $e \in E(G)$, $\varphi \in \mathcal{C}_k(G - e)$ where $k = \chi'(G) - 1 \geq \Delta + 1$, and

$$T = (y_0, e_1, y_1, \ldots, e_{n-1}, y_{n-1})$$

is a Tashkinov tree with respect to e and φ with $n = t(G)$.

Let $p = h(G)$. If $p < 5$ then, by Proposition 4.3, G is an elementary graph, implying $\chi'(G) = \mathcal{w}(G) \leq \tau_\epsilon(G)$, a contradiction. Consequently we have $p \geq 5$. Since (T, e, φ) is a balanced triple, we have $h(T) = p$, and Ty_{p-1} is the (α, β)-trunk of T with respect to φ. Hence, there are at most $\frac{n-p}{2} \geq \frac{n-5}{2}$ colours used on y_pT with respect to φ. Consequently, there are at most $\frac{n-5}{2} + 2 = \frac{n-1}{2}$ colours used on T with respect to φ.

From Theorem 3.3 we know that $V(T)$ is elementary and closed both with respect to φ. If $\Gamma^d(T, e, \varphi) = \emptyset$ then, by Proposition 3.4(c), $V(T)$ is also strongly closed with respect to φ, and therefore, by Theorem 2.8, G is an elementary graph, implying $\chi'(G) = \mathcal{w}(G) \leq \tau_\epsilon(G)$, a contradiction. Consequently, we have $\Gamma^d(T, e, \varphi) \neq \emptyset$. By Proposition 3.4(g), we also have $\Gamma^f(T, e, \varphi) \neq \emptyset$, implying $D(T, e, \varphi) \neq \emptyset$ and $F(T, e, \varphi) \neq \emptyset$.

Let $v \in D(T, e, \varphi)$. From Proposition 3.5 and Proposition 3.7 we conclude that $Y = V(T) \cup \{v\}$ is elementary with respect to φ. Then for a colour $\gamma \in \bar{\varphi}(V(T))$ there is an edge $f \in E_G(v)$ with $\varphi(f) = \gamma$. Since $V(T)$ is closed with respect to φ, the second endvertex v' of f belongs to $V(G) - V(T)$. Then $F = (f, v')$ is a fan at Y and therefore, by Theorem 3.10, $Z = Y \cup \{v'\}$ is elementary with respect to φ. Then Proposition 2.7 implies $n + 2 = |Z| \leq \frac{k-2}{k-\Delta} = 1 + \frac{\Delta-2}{k-\Delta}$. Moreover, from

$k+1 > \tau_\epsilon(G) \geq \lfloor (1+\epsilon)\Delta + 1 - 2\epsilon \rfloor$ we conclude $k-\Delta \geq \lfloor 1+\epsilon(\Delta-2) \rfloor \geq \epsilon(\Delta-2)$. Hence we have $n+2 \leq 1 + \frac{\Delta-2}{\epsilon(\Delta-2)} = 1 + \frac{1}{\epsilon}$ and, therefore, $n \leq \frac{1}{\epsilon} - 1$.

Since we have $F(T,e,\varphi) \neq \emptyset$, there is a vertex $u \in F(T,e,\varphi)$. Then, by Lemma 3.8, all colours from $\bar{\varphi}(u)$ are used on T with respect to φ. In the case $u \in \{y_0, y_1\}$ we have $|\bar{\varphi}(u)| = k - d_{G-e}(u) \geq k - \Delta + 1$ and hence at least $k - \Delta + 1$ colours are used on T with respect to φ. In the other case we have $u = y_j$ for some $j \in \{2, \ldots, n-1\}$ and $|\bar{\varphi}(u)| = k - d_{G-e}(u) \geq k - \Delta$. Then for the colour $\gamma = \varphi(e_j)$ we clearly have $\gamma \in \varphi(u)$ and, moreover, γ is used on T with respect to φ. Hence at least $k - \Delta + 1$ colours are used on T with respect to φ. Consequently, in both cases, at least $k - \Delta + 1$ colours and at most $\frac{n-1}{2}$ colours are used on T with respect to φ and, therefore, we conclude $k - \Delta + 1 \leq \frac{n-1}{2}$. Since $n \leq \frac{1}{\epsilon} - 1$, this implies $\chi'(G) = k + 1 \leq \Delta - 1 + \frac{1}{2\epsilon} \leq \tau_\epsilon(G)$, a contradiction. This completes the proof. ■

A consequence of Theorem 5.1 is the following result about an asymptotic approximation of the chromatic index for graphs with sufficiently large maximum degree. In particular, it supports Goldberg's conjecture asymptotically and extends a result of Kahn [5] (Theorem 2.2).

Corollary 5.2 *Let G be a graph with $\Delta(G) = \Delta$ and let $\epsilon > 0$.*

(a) *If $\Delta \geq \frac{1}{2\epsilon^2} - \frac{2}{\epsilon} + 2$ then*

$$\chi'(G) \leq \max\{(1+\epsilon)\Delta + 1 - 2\epsilon, w(G)\}.$$

(b) *If $\Delta \geq \frac{1}{2\epsilon^2}$ then*

$$\chi'(G) \leq \max\{(1+\epsilon)\Delta, w(G)\}.$$

Proof: Let $\Delta \geq \frac{1}{2\epsilon^2} - \frac{2}{\epsilon} + 2$. Then we have

$$\Delta + 1 + \epsilon(\Delta - 2) \geq \Delta + 1 + \epsilon\left(\frac{1}{2\epsilon^2} - \frac{2}{\epsilon}\right) = \Delta - 1 + \frac{1}{2\epsilon}$$

and, therefore,

$$\Delta + 1 + \epsilon(\Delta - 2) \geq \max\{\lfloor \Delta + 1 + \epsilon(\Delta - 2) \rfloor, \Delta - 1 + \frac{1}{2\epsilon}\}.$$

By Theorem 5.1, this implies

$$\chi'(G) \leq \tau_\epsilon(G) \leq \max\{(1+\epsilon)\Delta + 1 - 2\epsilon, w(G)\}.$$

Hence (a) is proved.

Now let $\Delta \geq \frac{1}{2\epsilon^2}$ and $\epsilon' = \frac{\epsilon}{1+2\epsilon}$. Then $\epsilon = \frac{\epsilon'}{1-2\epsilon}$ and, therefore,

$$\Delta \geq \frac{1}{2\epsilon^2} = \frac{(1-2\epsilon')^2}{2\epsilon'^2} = \frac{1}{2\epsilon'^2} - \frac{2}{\epsilon'} + 2.$$

From (a) we then infer $\chi'(G) \leq \max\{(1+\epsilon')\Delta + 1 - 2\epsilon', w(G)\}$. Moreover, we have

$$\begin{array}{rcl} (1+\epsilon')\Delta + 1 - 2\epsilon' & = & \Delta + \frac{\epsilon}{1+2\epsilon}\Delta + 1 - \frac{2\epsilon}{1+2\epsilon} \\ & = & \Delta + \frac{\epsilon}{1+2\epsilon}\Delta + \frac{1}{1+2\epsilon} \\ & \leq & \Delta + \frac{\epsilon}{1+2\epsilon}\Delta + \frac{2\epsilon^2\Delta}{1+2\epsilon} \\ & = & (1+\epsilon)\Delta. \end{array}$$

This proves (b). ■

The following result is a simple consequence from Corollary 5.2, matching the notation of the equivalent formulation of Goldberg's conjecture (Conjecture 2.3) and improving a result due to Favrholdt, Stiebitz and Toft [2].

Corollary 5.3 *Let G be a graph and $m \geq 3$ an odd integer. If $\Delta(G) \geq \frac{1}{2}(m-3)^2$ then*

$$\chi'(G) \leq \max\{\frac{m}{m-1}\Delta(G) + \frac{m-3}{m-1}, w(G)\}.$$

Proof: Let $\epsilon = \frac{1}{m-1}$. Then we have $\Delta(G) \geq \frac{1}{2}(\frac{1}{\epsilon} - 2)^2 = \frac{1}{2\epsilon^2} - \frac{2}{\epsilon} + 2$ and Corollary 5.2(a) implies $\chi'(G) \leq \max\{(1+\epsilon)\Delta + 1 - 2\epsilon, w(G)\} = \max\{\frac{m}{m-1}\Delta(G) + \frac{m-3}{m-1}, w(G)\}$. ■

Another consequence from Theorem 5.1 is an upper bound for χ' by means of its lower bound, the fractional chromatic index χ'_f.

Corollary 5.4 *Every graph G satisfies $\chi'(G) \leq \chi'_f(G) + \sqrt{\frac{1}{2}\chi'_f(G)}$.*

Proof: Let G be an arbitrary graph and $\Delta = \Delta(G)$. If $\Delta \leq 1$ then $E(G)$ is an, possibly empty, independent edge set, and we can colour all edges of G with the same colour. Obviously, we then have $\chi'(G) = \chi'_f(G) = \Delta$ and we are done. In the other case we have $\Delta \geq 2$ and, therefore,

$$\chi'_f(G) + \sqrt{\frac{1}{2}\chi'_f(G)} \geq \chi'_f(G) + 1 \geq w(G).$$

Moreover, for $\epsilon = \frac{1}{\sqrt{2\Delta}+2}$ we have

$$\begin{array}{rcl} (1+\epsilon)\Delta + 1 - 2\epsilon & = & \Delta + 1 + \epsilon(\Delta - 2) \quad = \quad \Delta + 1 + \frac{\Delta-2}{\sqrt{2\Delta}+2} \\ & = & \Delta + \frac{\Delta+\sqrt{2\Delta}}{\sqrt{2\Delta}+2} \quad = \quad \Delta + \sqrt{\frac{1}{2}\Delta} \\ & \leq & \chi'_f(G) + \sqrt{\frac{1}{2}\chi'_f(G)} \end{array}$$

as well as

$$\Delta - 1 + \tfrac{1}{2\epsilon} \;=\; \Delta + \sqrt{\tfrac{1}{2}\Delta} \;\leq\; \chi'_f(G) + \sqrt{\tfrac{1}{2}\chi'_f(G)}\ .$$

Hence we conclude $\tau_\epsilon(G) \leq \chi'_f(G) + \sqrt{\frac{1}{2}\chi'_f(G)}$. By Theorem 5.1, this completes the proof. ∎

6 Concluding Remarks

In Section 5 several upper bounds for the chromatic index χ' were developed. The question is whether they can be attained by an efficient algorithms. All these bounds are based on the results about Tashkinov trees in Section 4. Although the proofs of these results and all related proofs deal with optimal colourings in critical graphs, the used recolouring techniques work for arbitrary colourings in any graph. Based on these proofs several edge coloring algorithms can be constructed which will attain the several upper bounds. These algorithms will basically work by successively building and recolouring Tashkinov trees and will have time complexity polynomial in the numbers of vertices and edges. For the related proofs not in this paper the reader is referred to [2].

Since the Tashkinov tree is an improvement of the fan introduced by Vizing [13], this method of building and recolouring Tashkinov trees may be generally useful to construct new edge coloring algorithms or to improve various other that rely on Vinzing's fan argument.

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